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Mathematical analysis of equations describing the flow of compressible heat conducting fluids

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Abstract: The present thesis is devoted to the mathematical analysis of equations describing the flow of viscous compressible newtonian fluid in various time regimes. In particular, we present existence results for three problems arising as special cases of a general model derived in the introductory part. The first chapter deals with time-periodic solutions to the full Navier–Stokes–Fourier system for heat-conducting fluid. The second chapter contains the proof of existence of steady solutions to a system arising from phase field model for two-phase compressible fluid. Finally, in the last section we study steady strong solutions to the Navier–Stokes equations under the additional assumption that the fluid is sufficiently dense. For each problem a different concept of the solution is considered, on the other hand in all cases an essential role is played by the crucial quantity *effective viscous flux*.

Keywords: compressible Navier–Stokes system; weak solution; entropy variational solution; large data

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Contents

List of Symbols

$A \hookrightarrow B$	(continuous) embedding of A into B
$A \hookrightarrow\hookrightarrow B$	compact embedding of A into B
$\overline{g(\varrho, \mathbf{v}, c, \vartheta)}$	weak limit of a nonlinear expression g
$\{g\}_\Omega$	integral mean of g over Ω
$\nabla\Delta^{-1}$	inverse divergence, see equation (4.107)
α	coefficient of friction on the boundary
γ	adiabatic exponent
η	bulk viscosity coefficient, see equation (14)
ϵ, δ, ζ, τ	approximation parameters
ϑ	temperature
κ	heat conductivity coefficient
λ	viscosity coefficient, see equation (17)
μ	shear viscosity coefficient, see equation (14)
ϱ	density
σ	entropy production
χ_Ω	characteristic function of set Ω
χ_k	curvatures of boundary
$\boldsymbol{\omega}$	vorticity
Ω	domain
$\mathcal{B}$	Bogovskii's operator
c	relative concentration
$C(\Omega)$	space of continuous functions
$\overset{+}{c}$	concentration production
c_v	specific heat and constant volume
$C_{\mathrm{weak}}(S^1, X)$	space of weakly continuous functions
$\mathbb{D}(\mathbf{v})$	symmetric part of the velocity gradient
d	dimension, $d = 2, 3$

E	total energy density
e	internal energy
$\mathcal{E}$	total energy in Ω, see equation (1.57)
$\mathcal{F}$	Fourier transform
$\mathbf{g}$	external force density
G	effective viscous flux, see equation (2.25)
H	Helmholtz function
h	heat conductivity coefficient of boundary
$\mathbf{j}$	diffusive flux
$L^p(\Omega)$	Lebesgue space, see equation (4.100)
log	natural logarithm
m	mean density
$\mathcal{M}^+$	space of non-negative measures
M_0	total mass, see equation (1.46)
N	dimension of Galerkin approximation
$\mathbf{osc_q}$	oscillations defect measure, see equation (4.114)
p	pressure
$P_H,\ P_\nabla$	operators of Helmholtz decomposition, see equation (4.106)
$\mathbf{q}$	heat flux, see equation (15)
$\mathcal{Q}$	production of internal energy
$\mathcal{R}$	Riesz transform, see equation (4.108)
$\mathbb{S}$	viscous part of the stress tensor, see equation (14)
s	specific entropy density
S^1	time interval with periodic condition
$\mathbb{T}$	stress tensor, see equation (13)
$\mathbf{v}$	velocity field
$W^{m,p}(\Omega)$	Sobolev, or Sobolev-Slobodetskii space, see equation (4.101)
$W^{-1,p}(\Omega)$	dual Sobolev space, see equation (4.102)

Introduction

This dissertation deals with the mathematical analysis of equations modelling flows of viscous compressible fluids. We consider two time regimes, time-periodic and steady. Within the introductory part we present a general thermodynamically consistent model for two-phase viscous heat-conducting compressible newtonian fluid. The approach is based on inspecting the form of the rate of entropy production. The existence of solutions to special, simplified cases of the model are studied in three main chapters of the thesis.

In the first chapter, we consider the Navier–Stokes–Fourier system, which represents a well established and frequently studied model describing the flows of viscous compressible heat-conducting single constituted fluids. The existence of weak variational time-periodic solutions is proved. An important aspect of our investigation is, in contrast to the previous works, the presence of the radiation on the boundary, as well as inclusion of the non-compact boundary term into the entropy production. The shown results demonstrate the expected feature that the time-periodic regime lies indeed in-between the fully evolutionary and the steady case.

In the second chapter, we investigate the stationary compressible Navier–Stokes system coupled with the Allen–Cahn equation. Since only the isothermal situation is treated, we have in mind e.g. the process of melting or freezing at the level of almost constant critical temperature. We prove the existence of weak solutions, for heat capacity ratio $\gamma > 3$. For $\gamma > 6$ we establish the existence of solutions with point-wise bounded densities, which seems to be, according to famous Lions' counterexample [85], the best possible regularity of weak solutions, when we admit possible vacuum zones. We are aware of the fact that large γ's do not fit well the physical theory. However we can look at such result as an admissible approximation for the standard models.

The last chapter is focused on the steady Navier–Stokes equations for compressible fluid with density-dependent coefficients. The existence of the strong solutions is shown provided we are in the case of sufficiently large mass (keeping other data constant).

Although the results could seem to be at the first glance not to much connected to each other, there is definitely a common thread to all of them. Namely, the notion of effective viscous flux. This particular combination of unknowns plays really significant role for all concepts of the solutions considered, ranging from weak variational entropy solutions to strong ones.

Derivation of a general model

We will recall the derivation of a model of a heat-conducting viscous Newtonian compressible two-phase fluid in the framework of continuum fluid thermodynamics.[1]

[1] The possible applications for this kind of model are, as a matter of fact, restricted to certain length scale and amount of observable details. It is necessary to always keep in mind the limits of this description in any practical usage.

The state of the system is assumed to be fully characterised by the following set of the so called state variables: density ϱ, velocity field $\mathbf{v}$, relative concentration of one (selected) constituent c and absolute temperature ϑ. The other quantities are assumed to be either given as for the specific external forces $\mathbf{g}$ and the internal energy production $\mathcal{Q}$, or to be a functions of the state variables. This is the case of the diffusive flux $\mathbf{j}$, the production of the selected constituent $\overset{+}{c}$, the Cauchy stress tensor $\mathbb{T}$ together with the pressure p, the specific internal energy e, density of the entropy s, density of the total energy E and the heat flux $\mathbf{q}$. Last but not least we denote the specific entropy production by σ. In what follows we work in the Eulerian description, denoting $\dot{g}$ the material derivative of the quantity g, id est $\dot{g} = \frac{\partial g}{\partial t} + \mathbf{v} \cdot \nabla g$.

The basic physical principles are expressed in the terms of the balance laws, see Maršík [90]. The balance of mass takes the form

$$\dot{\varrho} = -\varrho \operatorname{div} \mathbf{v}, \tag{1}$$

the concentration of the phase can be changed either by a diffusive flux or by a phase production

$$\varrho \dot{c} = -\operatorname{div} \mathbf{j} + \overset{+}{c}. \tag{2}$$

The momentum equation representing the balance of the linear momentum reads

$$\varrho \dot{\mathbf{v}} = \operatorname{div} \mathbb{T} + \varrho \mathbf{g}, \tag{3}$$

while balance of the angular momentum is in the case of non-polar fluids reduced to the symmetry of the stress tensor

$$\mathbb{T} = \mathbb{T}^T. \tag{4}$$

Concerning the energy, let us first recall that from the momentum equation (3) the balance of the kinetic energy follows

$$\varrho \overline{(|\mathbf{v}|^2/2)}^{\cdot} = \operatorname{div}(\mathbb{T}\mathbf{v}) - \mathbb{T} : \nabla \mathbf{v} + \varrho \mathbf{g} \cdot \mathbf{v}. \tag{5}$$

Further, according to the first law of thermodynamics, the balance of the internal energy reads

$$\varrho \dot{e} = \mathbb{T} : \nabla \mathbf{v} - \operatorname{div} \mathbf{q} + \varrho \mathcal{Q}. \tag{6}$$

Summing up (5) and (6) we obtain the balance of the total energy

$$\varrho \dot{E} = \varrho \mathbf{v} \cdot \mathbf{g} + \operatorname{div}(\mathbb{T}\mathbf{v} - \mathbf{q}) + \varrho \mathcal{Q}, \tag{7}$$

where we have denoted the total energy

$$E = e + \frac{|\mathbf{v}|^2}{2}.$$

For simplicity, we will always neglect the heat sources $\mathcal{Q} \equiv 0$ as well as the diffusive flux $\mathbf{j} \equiv \mathbf{0}$. Under these assumptions we can deduce

$$\overline{\nabla c}^{\cdot} = -(\nabla \mathbf{v}) \nabla c + \nabla \left(\frac{\overset{+}{c}}{\varrho} \right). \tag{8}$$

In order to proceed, we will assume the internal energy to be of the form[2]

$$\varrho e = \varrho f(s, \varrho, c) + \nu(\varrho)\frac{|\nabla c|^2}{2}, \tag{9}$$

where the second term represents the energy of the diffuse interface. Taking the material derivative of (9) we get

$$\varrho \dot{e} = \varrho \frac{\partial e}{\partial s}\dot{s} + \left(\varrho\frac{\partial f}{\partial \varrho} + \frac{\nu'(\varrho)\varrho - \nu(\varrho)}{\varrho}\frac{|\nabla c|^2}{2}\right)\dot{\varrho} + \varrho\frac{\partial f}{\partial c}\dot{c} + \nu(\varrho)\nabla c\dot{\overline{\nabla c}}.$$

Denoting[3]

$$\vartheta = \frac{\partial e}{\partial s}, \qquad p = \varrho^2\frac{\partial f}{\partial \varrho}$$

we obtain with the use of the balance equations (1)–(8) the entropy balance

$$\begin{aligned}\vartheta\varrho\dot{s} = \big(\mathbb{T} + \nu(\varrho)\nabla c \otimes \nabla c\big) : \nabla \mathbf{v} + \left(p + \frac{\nu'(\varrho)\varrho - \nu(\varrho)}{2}|\nabla c|^2\right)\operatorname{div}\mathbf{v} \\ - \operatorname{div}\big(\mathbf{q} + \overset{+}{c}\nu(\varrho)\nabla c\big) + \overset{+}{c}\left(\frac{\operatorname{div}\big(\nu(\varrho)\nabla c\big)}{\varrho} - \frac{\partial f}{\partial c}\right).\end{aligned} \tag{10}$$

Hence after identifying the entropy flux,[4] we end up with the following relation for the production of the entropy,

$$\begin{aligned}\sigma = \varrho\dot{s} + \operatorname{div}\left(\frac{\mathbf{q} + \nu\nabla c\frac{\overset{+}{c}}{\varrho}}{\vartheta}\right) = \frac{1}{\vartheta}\Bigg(\mathbb{T} : \mathbb{D}(\mathbf{v}) + \left(p + \frac{\nu'(\varrho)\varrho - \nu(\varrho)}{2}|\nabla c|^2\right)\operatorname{div}\mathbf{v} \\ + \nu(\nabla c \otimes \nabla c) : \mathbb{D}(\mathbf{v}) - \mathbf{q}\frac{\nabla\vartheta}{\vartheta} + \overset{+}{c}\left(\frac{\operatorname{div}(\nu\nabla c)}{\varrho} - \frac{\nu\nabla c\cdot\nabla\vartheta}{\varrho\vartheta} - \frac{\partial f}{\partial c}\right)\Bigg),\end{aligned} \tag{11}$$

where we have replaced the full velocity gradient only by its symmetric part $\mathbb{D}(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} + \nabla^T\mathbf{v})$ due to the symmetry of the stress tensor (4). The second law of thermodynamics is fulfilled for

$$\sigma \geq 0. \tag{12}$$

Assuming linear dependences between the thermodynamical fluxes and thermodynamical affinities the following constitutive relations can be deduced[5]

$$\mathbb{T} = \mathbb{S}(\nabla\mathbf{v}, \varrho, c, \vartheta) - \left(\nu(\varrho)\nabla c \otimes \nabla c + \frac{\nu'(\varrho)\varrho - \nu(\varrho)}{2}|\nabla c|^2\,\mathbb{I}\right) - p(\varrho, c, \vartheta)\mathbb{I}, \tag{13}$$

$$\mathbb{S}(\nabla\mathbf{v}, \varrho, c, \vartheta) = 2\mu(\vartheta, \varrho, c)\left(\mathbb{D}(\mathbf{v}) - \frac{1}{d}\operatorname{div}\mathbf{v}\mathbb{I}\right) + \eta(\vartheta, \varrho, c)\operatorname{div}\mathbf{v}\mathbb{I}, \tag{14}$$

[2]We omit for simplicity the possible dependence of the gradient energy coefficient ν on temperature or concentration.

[3]Assuming $\vartheta > 0$ we can equivalently express the internal energy as a function of temperature instead of entropy, id est from now $e = e(\vartheta, \varrho, c, \nabla c)$

[4]Cf. also an alternative approach in [60].

[5]The same relations arise by postulating the constitutive relation for entropy production itself and the principle of maximal rate of entropy production, see e.g. Heida, Málek, Rajagopal [59, 88].

$$\mathbf{q} = -\kappa(\varrho, c, \vartheta)\frac{\nabla\vartheta}{\vartheta}, \tag{15}$$

$$\varrho\overset{+}{c} = C_0\Big(\operatorname{div}\big(\nu(\varrho)\nabla c\big) - \frac{\nu(\varrho)\nabla c\cdot\nabla\vartheta}{\vartheta} - \varrho\frac{\partial f}{\partial c}\Big). \tag{16}$$

Let us note that the crucial inequality (12) is satisfied provided

$$\mu \geq 0, \quad d\mu + 2\eta \geq 0, \quad \kappa \geq 0, \quad C_0 \geq 0.$$

In Chapters 2 and 3 it will be convenient to work with different notion of the second viscosity coefficient, namely

$$\lambda = \eta - \frac{2\mu}{d}, \tag{17}$$

formula (14) then reads

$$\mathbb{S}(\nabla\mathbf{v}, \varrho, c, \vartheta) = 2\mu\mathbb{D}(\mathbf{v}) + \lambda \operatorname{div}\mathbf{v}\mathbb{I}. \tag{18}$$

Moreover, we will always assume the fluid to be closed in a bounded container $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ with impermeable wall, so we prescribe the boundary conditions

$$\mathbf{v}\cdot\mathbf{n} = 0, \qquad \nabla c\cdot\mathbf{n} = 0 \quad \text{ on } \partial\Omega.$$

Thus, the complete system of partial differential equations, representing the balance laws together with our constitutive relations reads

$$\frac{\partial\varrho}{\partial t} + \operatorname{div}(\varrho\mathbf{v}) = 0, \tag{19}$$

$$\frac{\partial(\varrho c)}{\partial t} + \operatorname{div}(\varrho c\mathbf{v}) - \frac{C_0}{\varrho}\Big(\operatorname{div}\big(\nu\nabla c\big) - \frac{\nu\nabla c\cdot\nabla\vartheta}{\vartheta} - \varrho\frac{\partial f}{\partial c}\Big) = 0 \tag{20}$$

$$\begin{aligned}\frac{\partial(\varrho\mathbf{v})}{\partial t} &+ \operatorname{div}(\varrho\mathbf{v}\otimes\mathbf{v}) - \operatorname{div}\Big(2\mu\mathbb{D}(\mathbf{v}) - \nu\nabla c\otimes\nabla c\Big) \\ &+\nabla\Big(p + \frac{\varrho\nu' - \nu}{2}\,|\nabla c|^2 - \lambda\operatorname{div}\mathbf{v}\Big) - \varrho\mathbf{g} = \mathbf{0}\end{aligned} \tag{21}$$

$$\frac{\partial(\varrho s)}{\partial t} + \operatorname{div}(\varrho s\mathbf{v}) - \operatorname{div}\left(\frac{\kappa(\vartheta)\nabla\vartheta}{\vartheta^2}\right) = \sigma \tag{22}$$

$$\frac{\partial(\varrho E)}{\partial t} + \operatorname{div}(\varrho E\mathbf{v}) + \operatorname{div}\Big(\mathbb{T}\mathbf{v} + \frac{\kappa\nabla\vartheta}{\vartheta}\Big) - \varrho\mathbf{g}\cdot\mathbf{v} = 0 \tag{23}$$

with σ given by the right-hand side of (11), $\mathbb{T}$ from (13)–(14).

In literature, it is referred to this system usually in the following way. Taking into account only mechanical effects for single-constituted fluid, equations (13), (19) and (21) with $c \equiv const.$, $\vartheta \equiv const.$ are called the Navier–Stokes equations (NS). For $c \equiv const.$ the system is called Navier–Stokes–Fourier (NSF), and for varying c both temperature dependent and independent the Navier–Stokes–Allen–Cahn (NSAC). We will keep these notions.

Known results

In this section we would like to provide a brief overview of the most important developments yielding the state of art of the mathematical analysis of the compressible viscous flows. Especially, we try to concentrate on the existence results on bounded domains in two and three space dimensions for large data in the framework of classical Lebesgue and Sobolev spaces.

Pioneers

The very first rigorous result concerning the mathematical analysis of compressible Navier–Stokes equations was the proof of uniqueness of solutions with absence of the vacuum due to Serrin [137] and Graffi [58].

Later on, the first local-in-time existence results appeared. The classical solutions for full NSF in whole $\mathbb{R}^3$ were studied by Nash [108] and Itaya [68], see also Vol'pert, Hudjaev [151]. Concerning the corresponding results on bounded domains we refer to Solonnikov [141] and Valli [145] for NS and Tani [142] for NSF.

Particular contributions emerged in the case of the one-dimensional NS problem thanks to Kanel [75] Kazhikhov and Shelukhin [76], Padula [119], Antontsev et al. [4], Hoff [61, 62] or Serre [135].[6]

The first proof of the global-in-time existence of solutions to viscous compressible flows in $\mathbb{R}^3$ is due to Matsumura and Nishida [91] provided the initial data are sufficiently close to constant steady state with small external force, see also [92, 93] in the case of bounded and exterior domains. The stability of such solutions and its consequences was then studied by Valli [146–148], and Valli and Zajączkowski [149] for the case of heat-conducting fluids. These achievements were generalized for the case of large forces (but still small initial data) by Matsumura, Padula [94], Mucha and Zajączkowski [97, 105, 106].

Weak discontinuous solutions for small initial data were studied by Hoff [61, 63–66], Serre [135, 136] and Shelukhin [139]. Furthermore, Hoff and Serre [67] have shown in this framework the counterpart of the result of Serrin, id est possible non-uniqueness when the vacuum appears.

Concerning the existence of steady non-constant solutions near equilibrium, we refer to Padula [121], Beirão da Veiga [9] and Farwig [33, 34].

To conclude this subsection, Vaigant and Kazhikhov [144] presented the first correct[7] global existence result in multidimensional case for large initial data. More precisely, they showed the existence of classical and weak solutions on the square, provided a very special dependence of bulk viscosity on density.

Although a lot of the above mentioned articles contained innovative ideas, in order to get the desired global existence result in 3D without any artificial assumptions concerning the data, one needed to use more advanced mathematical tools.

[6] Much more recently the case of density dependent viscosity coefficients was studied as well, see e.g. [71, 86, 153] and references therein.

[7] Earlier, Padula [120] announced similar result in a more general setting, but there is an immovable gap in his proof.

Effective viscous flux identity, oscillation defect measure, and entropy methods

An importance of a special quantity - *effective viscous flux*, which possesses more compactness than its components, was observed already by Serre [136], Hoff [66] or Novotný [112], but it was Pierre-Louis Lions [82–84] who used it in order to make a real breakthrough in the mathematical analysis of compressible flows. In his celebrated monograph [85] he proved the global existence of the weak *renormalized*[8] solutions to NS (for $\gamma \geq \frac{9}{5}$ when $d = 3$, and for $\gamma > \frac{3}{2}$, $d = 2$), as well as the existence of steady solutions (see section below) for quite general data. Lions' eminent technique allows one to deal both whole $\mathbb{R}^d$ as well as domains with various boundary conditions; its main disadvantage lies, at least in the three-dimensional case, in the need of an artificially big value of the coefficient γ, which does not cover any physically relevant situation, and can be seen as a regularization of the original problem. Furthermore, only the case of constant viscosity coefficients was treated.

Both of the above mentioned disadvantages were, at least partially, overcome by Feireisl [35], who introduced the so-called *oscillations defect measure*, in order to obtain the compactness of the solutions in the case when the density is not a priori square integrable. Based on this observation, Feireisl, Novotný and Petzeltová [50] then showed the existence of global solutions to NS for $\gamma > \frac{d}{2}$,[9] and Feireisl [36] treated also the case of more general non-monotone pressure with the same growth. Later on, the existence of weak variational solution to NSF with constant viscosities for $\gamma = \frac{5}{3}$ was shown by Feireisl [38], see also monograph [37]. And finally, he generalized the previous results to non-constant viscosities [39, 40] in the framework of the so-called weak variational entropy solutions [45], see also monograph [48].

New original ideas came from discovering certain mathematical entropy *(BD entropy)*, first in the context of Korteweg fluids and shallow water by Bresch, Desjardins and Lin [11, 12, 15]. It relies on a specific differentiation relation between the two viscosity coefficients dependent on a density. These ideas were applied to obtain stability result for solutions to NS with density dependent viscosities by Mellet and Vasseur [95], see also [14], and to NSF in [13]. More recently a suitable approximative scheme was developed by Vasseur and Yu [150] as well.

The question of weak-strong uniqueness of quite regular solutions to NS (whose existence is not generally known) was addressed by Desjardins [23] and Germain [57]. Feireisl and Novotný [49] adapted the so-called *relative entropy method* in order to treat the same problem for weak variational entropy solutions to NSF as well. This method emerged to be extremely useful also when studying various singular limits of the full NSF, see e.g. [46–48]. Very recently, the method of relative entropy was used by Feireisl et al. [42] in order to show the existence and "weak–strong" uniqueness of so-called dissipative measure-valued solution for any $\gamma \geq 1$.

[8] This notion was introduced earlier by DiPerna and Lions [26] in the context of transport equation, see Section 4.6.

[9] Concerning the threshold for γ, the recent proof of existence for $\gamma = 1$ in 2D is due to Plotnikov and Weigant [128], see also an earlier work Erban [31] for a certain logarithmic growth of the pressure.

Steady solutions

Concerning the general large data steady solutions to the Navier–Stokes system for compressible fluids, in the pioneering work of Lions [85], the existence of weak variational solutions ($\gamma \geq \frac{5}{3}$ for $d = 3$; $\gamma > 1$ for $d = 2$) was proved, see also [118]. Subsequently, Novo and Novotný [110] modified the method of Feireisl in order to treat the lower adiabatic constants ($\gamma \leq \frac{5}{3}$), as soon as there are available corresponding a priori bounds.

The first ideas how to obtain these estimates proposed independently Frehse, Goj, Steinhauer [52], and Plotnikov, Sokolowski [125–127].

Březina and Novotný [18] showed the existence of steady solutions to three-dimensional NS for general volume forces when $\gamma > \frac{1+\sqrt{13}}{3}$, this restriction was further relaxed by Frehse, Steinhauer, Weigant [54] upto $\gamma > \frac{4}{3}$ (and $\gamma \geq 1$ in 2D, see [53]). Finally, using a unique bootstraping argument, Jiang, Zhou [73], Jesslé, Novotný [69] and Plotnikov, Weigant [129] reached in 3D the threshold $\gamma > 1$ in the case of periodic, slip and no-slip boundary conditions, respectively.

Concerning the large data steady solutions to three-dimensional NSF, the very first result is due to Mucha and Pokorný [101]. Their original proof relies on the fact that they work with the slip boundary condition, and it produces quite regular solutions with bounded density provided $\gamma > 3$. Later on [102], they modified it in order to cover $\gamma > \frac{7}{3}$ with both slip and no-slip boundary conditions. The same technique was used by Pecharová and Pokorný [122] to deal with $\gamma > 2$ in 2D.

Next, Novotný and Pokorný [115, 116] provided a further improvements of the allowed heat capacity ratio, which ensures the existence of weak variational entropy solution, up to $\gamma > \frac{3}{2}$, and $\gamma > \frac{3+\sqrt{41}}{8}$ respectively. Moreover, and in contrary to the previous results, they treated the case of temperature dependent viscosities. This method, when applied to the 2D problem yields the existence result with the condition $\gamma > 1$ or even with only a certain logarithmic growths, see [117, 131].

Finally, Jesslé, Novotný, Pokorný [70], showed the existence of weak variational entropy solution to NSF with slip boundary condition for velocity for any $\gamma > 1$, their solutions are weak as soon as $\gamma > \frac{5}{4}$. To conclude this subsection, let us mention the proof of existence of strong solutions for small Mach number [19, 27], and a recent investigation of stationary solutions to NS with inflow condition by Piasecki et al. [99, 123, 124]. See also review article [104].

Time-periodic solutions

The theory of the time-periodic solutions to the Navier–Stokes type systems for compressible fluids lies between the steady and fully evolutionary theory. To best our knowledge, time-periodic solutions to NS were first studied by Shelukhin [138] in one dimension; and by Valli [146] for sufficiently small data, see also Valli and Zajączkowski [149, Section 5] for NSF.

The existence of weak solution to the time-periodic NS for large data was shown by Feireisl et al. [43] for $\gamma > \frac{9}{5}$. The NSF system in the time-periodic setting was studied later on by Ma, Ukai, and Yong [87]. They proved the existence of classical solutions under some smallness assumptions for the case of rather unrealistic space dimension $d \geq 5$. Recently, the existence of the time-periodic weak variational entropy solutions for general data (see Section 1.1) was proved by Feireisl et al. [44]. However, only the case $\gamma = \frac{5}{3}$ was treated there, without the radiation on the boundary. The stability of the time-periodic compressible flows by spectral methods is studied by Březina, Kagei, Tsuda et al. [17, 74].

Navier–Stokes–Allen–Cahn system

The mathematical analysis of coupled systems: the compressible Navier–Stokes type and the phase separation is generally in its infancy [1, 24, 25, 51, 77, 78]. The existence of non-stationary weak solutions to isothermal NSAC with no-slip boundary condition for the velocity was shown by Feireisl et al. [51], where the hard sphere model for pressure was considered. Ding et al. [25] studied the global existence of weak, strong, and even classical solutions in 1D with free energy approximated by a suitable bistable function, assuming no vacuum zones in the initial data. Similar problem arising from modelling of stem cell differentiation in biological material in two dimensions was studied by Witterstein [152]. Further, Kotschote [77] put his attention to a more advanced model where the extra stress tensor is multiplied by density function ($\nu(\varrho) \sim \varrho$). He showed, however only, the local-in-time existence of strong solutions provided positiveness of the initial density, including the thermal effects as well. The existence of travelling waves for the corresponding isothermal model was shown by Freistühler [55].

1. Time-periodic solutions to the Navier–Stokes–Fourier system

In this chapter, we will consider a time-periodic flow of viscous heat-conducting single-constituted compressible fluid in a bounded domain described by NSF, id est we fix in system (19)–(23) $c \equiv const$. Moreover, we will take κ, μ, η independent of density. According to the introductory part we then obtain the following system of partial differential equations for unknowns ϱ, $\mathbf{v}$, and ϑ

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{v}) = 0, \tag{1.1}$$

$$\frac{\partial(\varrho \mathbf{v})}{\partial t} + \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v}) + \nabla p(\varrho, \vartheta) = \operatorname{div} \mathbb{S}(\vartheta, \nabla \mathbf{v}) + \varrho \mathbf{g}, \tag{1.2}$$

$$\frac{\partial\big(\varrho s(\varrho, \vartheta)\big)}{\partial t} + \operatorname{div}\big(\varrho s(\varrho, \vartheta)\mathbf{v}\big) + \operatorname{div}\left(\frac{\mathbf{q}(\vartheta, \nabla\vartheta)}{\vartheta}\right) = \sigma, \tag{1.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2}\varrho\, |\mathbf{v}|^2 + \varrho e(\varrho, \vartheta)\right) \mathrm{d}x = \int_\Omega \varrho \mathbf{g} \cdot \mathbf{v} \,\mathrm{d}x - \int_{\partial\Omega} \mathbf{q} \cdot \mathbf{n} \,\mathrm{d}S, \tag{1.4}$$

with

$$\mathbb{S}(\vartheta, \nabla \mathbf{v}) = \mu(\vartheta) \left(\nabla \mathbf{v} + \nabla^T \mathbf{v} - \frac{2}{d} \operatorname{div} \mathbf{v} \mathbb{I}\right) + \eta(\vartheta) \operatorname{div} \mathbf{v} \mathbb{I}, \tag{1.5}$$

$$\mathbf{q}(\vartheta, \nabla\vartheta) = -\kappa(\vartheta)\nabla\vartheta, \tag{1.6}$$

where the shear viscosity coefficient $\mu(\vartheta)$ is a globally Lipschitz function satisfying

$$0 < \underline{\mu}(1+\vartheta) \le \mu(\vartheta)$$

and the bulk viscosity coefficient $\eta(\vartheta)$ obeys[1]

$$0 \le \eta(\vartheta) \le \overline{\eta}(1+\vartheta).$$

The heat flux $\mathbf{q}$ fulfils Fourier's law with the heat conductivity coefficient $\kappa(\vartheta)$,

$$0 < \underline{\kappa}(1+\vartheta^3) \le \kappa(\vartheta) \le \overline{\kappa}(1+\vartheta^3).$$

All transport coefficients μ, η, κ are assumed to be continuously differentiable functions of ϑ.

The entropy production rate (11) is reduced to

$$\sigma = \frac{1}{\vartheta}\left(\mathbb{S}(\vartheta, \nabla\mathbf{v}) : \nabla\mathbf{v} - \frac{\mathbf{q}(\vartheta, \nabla\vartheta)}{\vartheta} \cdot \nabla\vartheta\right). \tag{1.7}$$

The fluid is contained in a smooth bounded domain Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$, we assume the following boundary conditions

$$\mathbf{v}|_{\partial\Omega} = \mathbf{0}, \tag{1.8}$$

$$\mathbf{q} \cdot \mathbf{n} = h(x, \vartheta)(\vartheta - \Theta_0), \tag{1.9}$$

[1] We are able to deal only with viscosities, which are not dependent explicitly on the density; this is crucial in the existence theory, and physically relevant at least for gases and certain liquids. Unfortunately, our choice does not include famous Sutherland's formula for ideal gases $\mu = \mu_0 \frac{\vartheta_0 + C}{\vartheta + C}\left(\frac{\vartheta}{\vartheta_0}\right)^{\frac{3}{2}}$.

where $0 < \underline{\Theta_0} \leq \Theta_0(x) \in L^1(\partial\Omega)$ represents the temperature of the boundary[2]. For the heat conductivity coefficient $h(x, \vartheta)$ we will consider two different cases:

$$
\begin{gathered}
h \text{ dependent on } \vartheta \text{ satisfying} \\
h_0(1+\vartheta^3) < h(x,\vartheta) < C(1+\vartheta^3),\ h_0 > 0, \\
h \text{ increasing and continuously differentiable with respect to } \vartheta, \\
h/\vartheta \text{ convex with respect to } \vartheta,
\end{gathered} \tag{1.10}
$$

$$
\begin{gathered}
h \text{ independent of } \vartheta \text{ satisfying} \\
0 < h_0 \leq h(x) \leq C < \infty.
\end{gathered} \tag{1.11}
$$

Let us note that according to The Second Law of Thermodynamics, the existence of non-trivial (changing in time) time-periodic flow within the energetically closed system is impossible. Hence a condition similar to (1.9) which admits a heat flux through boundary is actually necessary to have the opportunity to get a non-trivial solution to our problem. The fluid is driven by a time-periodic external force with a given period $L > 0$

$$
\mathbf{g} \in L^\infty(\mathbb{R}^1 \times \Omega, \mathbb{R}^d),\ \mathbf{g}(t+L,\cdot) = \mathbf{g}(t,\cdot),\ \forall t \in \mathbb{R}.
$$

The thermodynamical quantities: the pressure p, the specific entropy s, and the specific internal energy e are specified so that they satisfy the Gibbs relation

$$
\vartheta Ds(\varrho,\vartheta) = De(\varrho,\vartheta) + p(\varrho,\vartheta)D(1/\varrho). \tag{1.12}
$$

Although we will use more general structure assumption for p, s, and e, we will always have in mind the prototype dependence

$$
p(\varrho,\vartheta) = \varrho^\gamma + \varrho\vartheta + \frac{a}{3}\vartheta^4, \tag{1.13}
$$

$$
e(\varrho,\vartheta) = \frac{1}{\gamma-1}\varrho^{\gamma-1} + c_v\vartheta + \frac{a}{\varrho}\vartheta^4, \tag{1.14}
$$

$$
s(\varrho,\vartheta) = \log\left(\frac{\vartheta^{c_v}}{\varrho}\right) + \frac{4a}{3\varrho}\vartheta^3, \tag{1.15}
$$

where a, γ, and c_v are positive constants.[3] The possible values of γ required for the existence results will be specified later. Following [48] we assume the pressure to be of the form

$$
p(\varrho,\vartheta) = \vartheta^{\frac{\gamma}{\gamma-1}} P\left(\frac{\varrho}{\vartheta^{\frac{1}{\gamma-1}}}\right) + \frac{a}{3}\vartheta^4 \tag{1.16}
$$

where $P \in C^1[0,\infty) \cap C^2(0,\infty)$, with $P'(z) > 0$ for all $z \geq 0$, satisfying

$$
\lim_{z\to\infty} \frac{P'(z)}{z^{\gamma-1}} = p_\infty > 0. \tag{1.17}
$$

[2]We could allow the function $\Theta_0(x,t)$ to be dependent on time in a time-periodic way as well, but we omit it.

[3]Here, and in all what follows, log denotes the natural logarithm.

In order to satisfy the Gibbs relation we can take[4]

$$e(\varrho,\vartheta) = \frac{1}{\gamma-1}\,\frac{\vartheta^{\frac{\gamma}{\gamma-1}}}{\varrho}\,P\Big(\frac{\varrho}{\vartheta^{\frac{1}{\gamma-1}}}\Big) + \frac{a}{\varrho}\vartheta^4,$$

$$s(\varrho,\vartheta) = S\Big(\frac{\varrho}{\vartheta^{\frac{1}{\gamma-1}}}\Big) + \frac{4a}{3\varrho}\vartheta^3, \tag{1.18}$$

with

$$S'(z) = -\frac{1}{\gamma-1}\frac{\gamma P(z) - zP'(z)}{z^2}. \tag{1.19}$$

Finally, we assume specific heat at constant volume to be bounded and positive

$$0 < \frac{\gamma P(z) - zP'(z)}{z} \le C. \tag{1.20}$$

It can be deduced from (1.16)–(1.19) that we have the form

$$p(\varrho,\vartheta) = p_0(\varrho,\vartheta) + \frac{a}{3}\vartheta^4, \tag{1.21}$$

$$e(\varrho,\vartheta) = e_0(\varrho,\vartheta) + \frac{a}{\varrho}\vartheta^4, \tag{1.22}$$

$$s(\varrho,\vartheta) = s_0(\varrho,\vartheta) + \frac{4a}{3\varrho}\vartheta^3, \tag{1.23}$$

where

$$c\varrho^\gamma \le p_0(\varrho,\vartheta) \le C(\varrho\vartheta + \varrho^\gamma), \tag{1.24}$$

$$c\varrho^\gamma \le \varrho e_0(\varrho,\vartheta) \le C(\varrho\vartheta + \varrho^\gamma). \tag{1.25}$$

Extracting the main part of the cold pressure, we can decompose the pressure further as follows

$$p_0(\varrho,\vartheta) = c\varrho^\gamma + p_m(\varrho,\vartheta), \text{ with } c>0 \text{ and } \frac{\partial p_m}{\partial\varrho} \ge 0. \tag{1.26}$$

Moreover,

$$\frac{\partial p(\varrho,\vartheta)}{\partial\varrho} > 0, \qquad 0 < \frac{\partial e_0(\varrho,\vartheta)}{\partial\vartheta} \le C, \qquad \frac{\partial e_0(\varrho,\vartheta)}{\partial\varrho}\varrho \le C(\varrho^{\gamma-1} + \vartheta). \tag{1.27}$$

Finally, since $s_0(\varrho,\vartheta)$ is increasing in ϑ and decreasing in ϱ

$$\frac{\partial s_0(\varrho,\vartheta)}{\partial\varrho} < 0, \qquad \frac{\partial s_0(\varrho,\vartheta)}{\partial\vartheta} = \frac{1}{\vartheta}\frac{\partial e_0(\varrho,\vartheta)}{\partial\vartheta} > 0, \tag{1.28}$$

we obtain (for suitable additive constant) by expressing

$$s_0(\varrho,\vartheta) = s_0(\varrho,1) + \int_1^\vartheta \frac{\partial s_0(\varrho,\tau)}{\partial\vartheta}\mathrm{d}\tau$$

---

[4]Indeed,

$$\mathrm{d}e = \Big[\frac{\vartheta^{\frac{\gamma}{\gamma-1}}}{\gamma-1}P\Big(\frac{\varrho}{\vartheta^{\frac{1}{\gamma-1}}}\Big) - \frac{\vartheta\varrho}{\gamma-1}P'\Big(\frac{\varrho}{\vartheta^{\frac{1}{\gamma-1}}}\Big) + a\vartheta^4\Big]\mathrm{d}\Big(\frac{1}{\varrho}\Big) + \Big[-\frac{\varrho}{(\gamma-1)\vartheta^{\frac{\gamma}{\gamma-1}}}S'\Big(\frac{\varrho}{\vartheta^{\frac{1}{\gamma-1}}}\Big) + \frac{4a}{\varrho}\vartheta^4\Big]\mathrm{d}\vartheta$$

that

$$
\begin{aligned}
|s_0(\varrho,\vartheta)| &\le C(1+|\log\varrho|+|\log\vartheta|) && \text{in } (0,\infty)^2, && (1.29)\\
|s_0(\varrho,\vartheta)| &\le C(1+|\log\varrho|) && \text{in } (0,\infty)\times(1,\infty), && (1.30)\\
s_0(\varrho,\vartheta) &\ge c>0 && \text{in } (0,1)\times(1,\infty), && (1.31)\\
s_0(\varrho,\vartheta) &\ge c(1+\log\vartheta) && \text{in } (0,1)\times(0,1). && (1.32)
\end{aligned}
$$

The chapter is organised in the following way. Firstly, we will introduce the concept of the weak variational entropy solution and present our main results. Secondly, we will show the a priori estimates for the solutions on purely heuristic level in four cases under consideration. This is motivation for our definition of weak solutions, and at the same time it is the core of the technical part of the proofs, to which the rest of the chapter is devoted. We will introduce the approximation scheme, show the existence of approximative solutions and then pass to the limit.

1.1 Definition of the solutions, the main results

As we search for the time-periodic solutions with period L, we will consider all quantities defined on the time interval $S^1 = [0,L]|_{\{0,L\}}$, accompanied with the periodicity condition

$$g(0,.) = g(L,.).$$

We call a triple $\{\varrho,\mathbf{v},\vartheta\}$ a time-periodic weak variational entropy solution to the Navier–Stokes–Fourier system, if the following holds true[5]

$$\varrho \ge 0, \quad \vartheta > 0 \text{ almost everywhere,} \tag{1.33}$$

$$\varrho \in L^\infty\left(S^1; L^\gamma(\Omega)\right), \ \mathbf{v} \in L^2(S^1; W_0^{1,2}(\Omega;\mathbb{R}^d)), \tag{1.34}$$

$$\vartheta \in L^\infty\left(S^1; L^4(\Omega)\right), \quad \vartheta^{3/2}, \log\vartheta \in L^2(S^1; W^{1,2}(\Omega)) \tag{1.35}$$

$$\Theta_0/\vartheta \in L^1(S^1\times\partial\Omega), \tag{1.36}$$

$$\vartheta \in L^4(S^1\times\partial\Omega), \text{ or } \vartheta \in L^1(S^1\times\partial\Omega), \text{ respectively.} \tag{1.37}$$

Moreover,

$$\varrho \in C_{\mathrm{weak}}(S^1, L^\gamma(\Omega)), \tag{1.38}$$

$$\varrho\mathbf{v} \in C_{\mathrm{weak}}(S^1; L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^3)). \tag{1.39}$$

The continuity equation is satisfied in the renormalized sense, id est, for all $b \in C^1[0,\infty)$, and any $\psi \in C^\infty(S^1\times\overline{\Omega})$

$$\int_{S^1}\int_\Omega \left(b(\varrho)\frac{\partial\psi}{\partial t} + b(\varrho)\mathbf{v}\cdot\nabla\psi + \big(b(\varrho)-b'(\varrho)\varrho\big)\operatorname{div}\mathbf{v}\psi \right) \mathrm{d}x\,\mathrm{d}t = 0, \tag{1.40}$$

the momentum equation is satisfied in the sense of distributions, id est, for all $\boldsymbol{\varphi} \in C^\infty(S^1\times\Omega,\mathbb{R}^d)$ with $\boldsymbol{\varphi} = \mathbf{0}$ at $\partial\Omega$

$$
\begin{aligned}
\int_{S^1}\int_\Omega \left(\varrho\mathbf{v}\frac{\partial\boldsymbol{\varphi}}{\partial t} + (\varrho\mathbf{v}\otimes\mathbf{v}) : \nabla\boldsymbol{\varphi} + p(\varrho,\vartheta)\operatorname{div}\boldsymbol{\varphi} \right) \mathrm{d}x\,\mathrm{d}t \\
= \int_{S^1}\int_\Omega \left(\mathbb{S}(\vartheta,\nabla\mathbf{v}) : \nabla\boldsymbol{\varphi} - \varrho\mathbf{g}\cdot\boldsymbol{\varphi} \right) \mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{1.41}
$$

[5]Note we are not able to exclude possible vacuum areas.

the specific entropy satisfies for all $\psi \in C^\infty(S^1 \times \overline{\Omega})$

$$\int_{S^1}\int_\Omega \left(\varrho s(\varrho,\vartheta)\frac{\partial \psi}{\partial t} + \varrho s(\varrho,\vartheta)\mathbf{v}\cdot\nabla\psi + \frac{\mathbf{q}(\vartheta,\nabla\vartheta)}{\vartheta}\cdot\nabla\psi\right)\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{S^1}\int_{\partial\Omega} h(x,\vartheta)\psi\,\mathrm{d}S\,\mathrm{d}t - \langle \sigma,\psi\rangle\,, \quad (1.42)$$

where the production of entropy σ is represented by a non-negative measure $\sigma \in \mathcal{M}^+(S^1\times\overline{\Omega})$ satisfying for any non-negative $\psi \in C^\infty(S^1\times\overline{\Omega})$ and a.a. $t \in S^1$

$$\langle\sigma,\psi\rangle \geq \int_\Omega \frac{1}{\vartheta}\left(\mathbb{S}(\vartheta,\nabla\mathbf{v}):\nabla\mathbf{v} - \frac{\mathbf{q}(\vartheta,\nabla\vartheta)}{\vartheta}\cdot\nabla\vartheta\right)\psi\,\mathrm{d}x + \int_{\partial\Omega} h(x,\vartheta)\frac{\Theta_0}{\vartheta}\psi\,\mathrm{d}S, \quad (1.43)$$

and the balance of total energy, for all $\psi \in C^\infty(S^1)$

$$\int_{S^1}\left(\frac{\partial\psi}{\partial t}\int_\Omega\left(\frac{1}{2}\varrho\,|\mathbf{v}|^2 + \varrho e(\varrho,\vartheta)\right)\mathrm{d}x\right)\mathrm{d}t$$
$$= \int_{S^1}\psi\Big(\int_{\partial\Omega} h(x,\vartheta)(\vartheta-\Theta_0)\,\mathrm{d}S - \int_\Omega \varrho\mathbf{g}\cdot\mathbf{v}\,\mathrm{d}x\Big)\,\mathrm{d}t. \quad (1.44)$$

Remark 1.1. Recall that the equations of fluid thermodynamics are often formulated as balance of mass, momentum and internal (or total) energy, id est, equation (1.3) is replaced either by (6) or (7). These equations are (in the weak sense) equivalent, provided one can use as test function in the momentum equation the velocity $\mathbf{v}$. However, in our case, we are not able to control the convective term in the total energy balance $\varrho|\mathbf{v}|^2\mathbf{v}$; even for large γ we would not be able to ensure time compactness of this term. Therefore we cannot verify the validity of the total energy balance. Concerning the internal energy balance, the limiting term is $\mathbb{T}:\nabla\mathbf{v}$, in which one can not pass to the limit. Therefore we have to use, similarly as for the initial value problem, the variational entropy formulation. As a matter of fact, since in our definition of solution the equality (1.7) is replaced by inequality (1.43), we allow our solutions to produce more entropy than expected. On the other hand, this inequality is somehow compensated by the equality in the integrated total energy balance (1.44), so we are still able to show that the variational entropy solutions coincides with the classical ones as soon as they are smooth. This concept of solutions is in the spirit of weak solutions with defect measure introduced by DiPerna and Lions [26] in the context of transport equations, cf. Bresch and Desjardins [13, page 9], who pointed out that *"no discontinuities are expected in the viscous and heat conducting case"*.

We study the existence of above-defined time-periodic weak variational entropy solutions with a given period L and improve the result of Feireisl et al. [44] in the following sense: we extend the class of pressure functions (i.e. consider lower exponent γ) and include also the effect of radiation on the boundary, in both 2D and 3D case. The results of this chapter are contained in a series of articles [6, 8].

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a $C^{2+\nu}$ boundary. Assume that the above mentioned hypotheses are all satisfied with h dependent on ϑ satisfying (1.10), $\Theta_0 \in L^4(\partial\Omega)$, and $\gamma > \gamma_0$, where*

$$\gamma_0 = \begin{cases} 1, & d = 2, \\ \frac{23}{15}, & d = 3. \end{cases} \tag{1.45}$$

Then for any $M_0 > 0$ there exists at least one time-periodic weak variational entropy solution to the Navier–Stokes–Fourier system such that

$$\int_\Omega \varrho(t, \cdot)\, \mathrm{d}x = M_0. \tag{1.46}$$

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a $C^{2+\nu}$ boundary. Assume that the above mentioned hypotheses are all satisfied with h independent of ϑ satisfying (1.11) and $\gamma > \gamma_0$, where*

$$\gamma_0 = \begin{cases} 1, & d = 2, \\ \frac{8}{5}, & d = 3. \end{cases} \tag{1.47}$$

Then for any $M_0 > 0$ there exists at least one variational entropy time-periodic solution to the Navier–Stokes–Fourier system such that

$$\int_\Omega \varrho(t, \cdot)\, \mathrm{d}x = M_0. \tag{1.48}$$

We will present here in all details only the proof of Theorem 1.2 in 3D case, since the proof of the 2D case is easier and the proof of Theorem 1.3 closely follows the former article [44]. The central point are the a priori bounds which will be given in all cases, exposing the main differences of four situations under consideration.

Remark 1.4. Note $\frac{3}{2} < \frac{23}{15} < \frac{8}{5} < \frac{5}{3}$, thus in both cases we deal with more general pressure laws than the aforementioned result [44], and further, in the model with radiation on the boundary we are only $\frac{1}{30}$ above the "optimal" exponent[6] $\gamma = \frac{3}{2}$.

Remark 1.5. The heat flux $\mathbf{q}$ satisfies the Fourier law $\mathbf{q}(\vartheta, \nabla\vartheta) = -\kappa(\vartheta)\nabla\vartheta$, with the heat conductivity coefficient $\kappa(\vartheta)$, $0 < \underline{\kappa}(1 + \vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1 + \vartheta^3)$ taking into account the Stefan–Boltzmann type radiation, therefore it is natural to take analogous condition also on the boundary. From purely mathematical point of view, the advantage of this choice is that we are able to deduce better time integrability of the temperature on the boundary ($\vartheta \in L^4(S^1 \times \partial\Omega)$), and consequently also inside the domain due to the Poincaré inequality. On the other hand, we will have to identify the limit for the additional non-linearity in this model. Note that Ducomet and Feireisl [28] observed similar effect inside the domain.

[6]For lower exponents we are not able to bound the kinetic energy in Bogovskii estimates and thus to give a sense to the convective term.

1.2 A priori bounds

Throughout the thesis we will denote all generic constants by C, they may differ from line to line, or even in the same formula, and they can depend on controlled norms of given data.

1.2.1 A priori bounds without radiation on the boundary

Energy estimates

Without loss of generality, we will consider only $\gamma \in (\gamma_0, 2)$; the cases $\gamma \geq 2$ are much easier and dealt in article [44]. Our first observation is that the conservation of mass (1.1) and (1.46) yields

$$\varrho \in L^\infty\left(S^1; L^1(\Omega)\right). \tag{1.49}$$

Further, we can put $\psi \equiv 1$ in (1.42) in order to get

$$\int_{S^1}\int_\Omega \left(\frac{\mathbb{S}(\vartheta, \nabla \mathbf{v}) : \nabla \mathbf{v}}{\vartheta} - \frac{\mathbf{q}(\vartheta, \nabla\vartheta)}{\vartheta^2}\cdot\nabla\vartheta\right) \mathrm{d}x\,\mathrm{d}t \leq \int_{S^1}\int_{\partial\Omega} \frac{h(x)(\vartheta - \Theta_0)}{\vartheta}\,\mathrm{d}S\,\mathrm{d}t. \tag{1.50}$$

Note $\frac{\mathbb{S}(\vartheta,\nabla\mathbf{v}):\nabla\mathbf{v}}{\vartheta} \geq \frac{\underline{\mu}(1+\vartheta)}{\vartheta}\left|\nabla\mathbf{v} + \nabla^T\mathbf{v} - \frac{2}{d}\operatorname{div}\mathbf{v}\mathbb{I}\right|^2 \geq \underline{\mu}\left|\nabla\mathbf{v} + \nabla^T\mathbf{v} - \frac{2}{d}\operatorname{div}\mathbf{v}\mathbb{I}\right|^2$, so plugging the form of $\mathbf{q}$ into the inequality as well and applying Korn's inequality from the first part of Theorem 4.4 yields

$$\int_{S^1}\int_\Omega \left(C\left|\nabla\mathbf{v}\right|^2 + \frac{\kappa(\vartheta)\left|\nabla\vartheta\right|^2}{\vartheta^2}\right)\mathrm{d}x\,\mathrm{d}t + \int_{S^1}\int_{\partial\Omega}\frac{h\Theta_0}{\vartheta}\,\mathrm{d}S\,\mathrm{d}t \leq \int_{S^1}\int_{\partial\Omega} h\,\mathrm{d}S\,\mathrm{d}t.$$

Consequently, since $\frac{\kappa(\vartheta)}{\vartheta^2}\left|\nabla\vartheta\right|^2 \geq \underline{\kappa}\frac{(\vartheta^3+1)}{\vartheta^2}\left|\nabla\vartheta\right|^2 = \underline{\kappa}\left((\vartheta^{1/2}+\vartheta^{-1})\left|\nabla\vartheta\right|\right)^2$, we have

$$\mathbf{v} \in L^2\big(S^1, W_0^{1,2}(\Omega)\big), \tag{1.51}$$

$$\nabla(\vartheta^{3/2}),\ \nabla(\log\vartheta) \in L^2\left(S^1; L^2(\Omega)\right),\ \frac{\Theta_0}{\vartheta} \in L^1(S^1\times\Omega). \tag{1.52}$$

Taking $\psi \equiv 1$ in (1.44), we deduce for $\frac{1}{r}+\frac{1}{q} = 1$ that

$$\int_{S^1}\int_{\partial\Omega} h(\vartheta - \Theta_0)\,\mathrm{d}S\,\mathrm{d}t = \int_{S^1}\int_\Omega \varrho\mathbf{g}\cdot\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t \leq C\left\|\varrho\right\|_{L^2(L^r)}\left\|\mathbf{v}\right\|_{L^2(L^q)}. \tag{1.53}$$

We have $W_0^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ with $q = 6$ in 3D and arbitrary $q < \infty$ in 2D, hence according to the properties of h, and Θ_0

$$\left\|\vartheta\right\|_{L^1(S^1\times\partial\Omega)} \leq C(r)\big(1 + \left\|\varrho\right\|_{L^2(L^r)}\big), \tag{1.54}$$

provided $r = \frac{6}{5}$, $r > 1$ respectively. This yields together with the basic entropy estimate and the Poincaré inequality for q as above

$$\left\|\vartheta\right\|_{L^1\left(L^{3q/2}\right)} \leq C(r,q)\big(1 + \left\|\varrho\right\|_{L^2(L^r)}\big), \tag{1.55}$$

moreover by a simple interpolation with $\left\|\varrho\right\|_{L^\infty(L^1)}$ we can bound

$$\left\|\varrho\right\|_{L^2(L^r)} \leq C\left(1 + \left(\int_{S^1}\left(\int_\Omega \varrho^\gamma\,\mathrm{d}x\right)^{\frac{2(r-1)}{(\gamma-1)r}}\mathrm{d}t\right)^{1/2}\right). \tag{1.56}$$

Note that $\frac{2(r-1)\gamma}{(\gamma-1)r} < \gamma$ for $r < \frac{2}{3-\gamma}$, which is satisfied in 3D for $\gamma > 4/3$ so in fact in this case

$$\|\varrho\|_{L^2(L^r)} \le C \left(1 + \|\varrho\|_{L^\gamma(L^\gamma)}^{\frac{(r-1)\gamma}{(\gamma-1)r}} \right). \tag{1.57}$$

Denoting the total (integrated) energy by

$$\mathcal{E}(t) = \int_\Omega \varrho E(t)\,\mathrm{d}x = \int_\Omega \left(\frac{1}{2}\varrho\,|\mathbf{v}|^2 + \varrho e(\varrho,\vartheta)\right)\mathrm{d}x,$$

we get from its balance (1.44), with use of (1.22) and (1.25), and combining (1.53) and (1.56) that for all $t_1,\ t_2 \in S^1$

$$\begin{aligned}\mathcal{E}(t_1) - \mathcal{E}(t_2) &\le C\left(1 + \int_{S^1} \mathcal{E}(t)\,\mathrm{d}t\right),\\ \sup_{t\in S^1} \mathcal{E}(t) &\le C\left(1 + \int_{S^1} \mathcal{E}(t)\,\mathrm{d}t\right).\end{aligned}$$

If we use the information which comes from (1.49) and (1.51), we can bound the kinetic energy

$$\int_{S^1}\int_\Omega \frac{1}{2}\varrho\,|\mathbf{v}|^2\,\mathrm{d}x\,\mathrm{d}t \le C\,\|\varrho\|_{L^\infty(L^r)} \le \varepsilon \sup_{t\in S^1}\int_\Omega \varrho e(\varrho,\vartheta)\,\mathrm{d}x + C_\varepsilon(M_0), \tag{1.58}$$

whence for the total energy we obtain

$$\sup_{t\in S^1}\mathcal{E}(t) \le C\left(1 + \int_{S^1}\int_\Omega \varrho e(\varrho,\vartheta)\,\mathrm{d}x\,\mathrm{d}t\right) \le C\left(1 + \int_{S^1}\int_\Omega \left(\varrho^\gamma + \varrho\vartheta + \vartheta^4\right)\mathrm{d}x\,\mathrm{d}t\right). \tag{1.59}$$

The first term on the right-hand side will be left as it is. The last term will be estimated as follows

$$\|\vartheta\|_{L^4}^4 \le \|\vartheta\|_{L^4}\,\|\vartheta\|_{L^4}^3 \le \|\vartheta\|_{L^4} \sup_{t\in S^1}\mathcal{E}^{3/4}(t), \tag{1.60}$$

which yields in 2D using the bounds (1.55) and (1.56)

$$\begin{aligned}\int_{S^1}\int_\Omega \vartheta^4\,\mathrm{d}x\,\mathrm{d}t &\le \|\vartheta\|_{L^1(L^4)} \sup_{t\in S^1}\mathcal{E}^{3/4}(t) \le C(1 + \|\varrho\|_{L^2(L^r)}) \sup_{t\in S^1}\mathcal{E}^{3/4}(t)\\ &\le C\left(1 + \int_{S^1}\int_\Omega \varrho^\gamma\,\mathrm{d}x\,\mathrm{d}t\right)^{1/4} \sup_{t\in S^1}\mathcal{E}^{3/4}(t).\end{aligned} \tag{1.61}$$

Further, by virtue of (1.55) we estimate the remaining term

$$\begin{aligned}\int_{S^1}\int_\Omega \varrho\vartheta\,\mathrm{d}x\,\mathrm{d}t &\le \|\varrho\|_{L^\infty(L^\gamma)}\,\|\vartheta\|_{L^1\left(L^{\gamma/\gamma-1}\right)}\\ &\le C\,\|\varrho\|_{L^\infty(L^\gamma)}\,\|\varrho\|_{L^2(L^r)} \le C \sup_{t\in S^1}\mathcal{E}(t)^{1/\gamma}\,\|\varrho\|_{L^2(L^r)}.\end{aligned} \tag{1.62}$$

Our further considerations will lead to the fact that ϱ is bounded in $L^{a\gamma}(S^1\times\Omega)$, with $a = 1 + \frac{\gamma-s}{\gamma s}$, where $s = \frac{\gamma+1}{2}$ in 2D, and $s = \frac{3}{2}$ in 3D. Therefore, we interpolate for $r \in (1, \gamma + \frac{\gamma-s}{s})$ as follows

$$\|\varrho\|_{L^2(L^r)} \le \|\varrho\|_{L^\infty(L^1)}^{1-\alpha}\,\|\varrho\|_{L^{\gamma+\frac{\gamma-s}{s}}(S^1\times\Omega)}^{\alpha},$$

with $\alpha = \frac{r-1}{r} \cdot \frac{\gamma s+\gamma-s}{\gamma s+\gamma-2s}$, so we get using Young's inequality

$$\sup_{t\in S^1} \mathcal{E}(t) \leq C\left(1 + \|\varrho\|^{\gamma}_{L^\gamma(S^1\times\Omega)} + \|\varrho\|^{\alpha\gamma/\gamma-1}_{L^{\gamma+\frac{\gamma-s}{s}}(S^1\times\Omega)}\right). \tag{1.63}$$

In order to finish the estimates, we have to ensure that the power of the last term on the right-hand side is less than $a\gamma$, id est

$$\alpha \cdot \frac{\gamma}{\gamma-1} < \gamma + \frac{\gamma - s}{s}. \tag{1.64}$$

This can be satisfied in 2D due to the fact that r can be chosen arbitrarily close to one, for example for the choice $r := (\gamma s + \gamma - 2s)(\gamma-1)^2 + 1$. In 3D we have inequality $\varrho\vartheta \leq C(\varrho^\gamma + 1 + \vartheta^4)$, so more restrictive is the last term, namely

$$\frac{4a\gamma}{6(a\gamma-1)} < a\gamma, \tag{1.65}$$

which gives us the constraint $a\gamma > \frac{5}{3}$, and accordingly (as $a = \frac{5\gamma-3}{3\gamma}$)

$$\gamma > \frac{8}{5}. \tag{1.66}$$

Pressure estimates

It remains to deduce suitable estimates of density, this will be done by testing the momentum equation (1.41) with[7]

$$\mathbf{\Phi} = \mathcal{B}\left[\varrho^{\gamma(a-1)} - \left\{\varrho^{\gamma(a-1)}\right\}_\Omega\right],$$

where a is as above[8], and $\mathcal{B} \sim \operatorname{div}^{-1}$ is the Bogovskii operator. From its properties and the conservation of mass, see Theorem 4.13, we can deduce that $\mathbf{\Phi} \in L^\infty\left(S^1, W^{\frac{1}{\gamma(a-1)}}(\Omega)\right)$, and $\left\{\varrho^{\gamma(a-1)}\right\}_\Omega \in L^\infty(S^1)$, so it follows from (1.24) that

$$\begin{aligned}
&\int_{S^1}\int_\Omega \left(p(\varrho,\vartheta)\varrho^{\gamma(a-1)}\right) \mathrm{d}x\,\mathrm{d}t \\
&\quad\leq \int_{S^1}\int_\Omega \left(-\varrho\mathbf{v}\frac{\partial\mathbf{\Phi}}{\partial t} - (\varrho\mathbf{v}\otimes\mathbf{v}) : \nabla\mathbf{\Phi} + \mathbb{S}(\vartheta,\nabla\mathbf{v}) : \nabla\mathbf{\Phi} - \varrho\mathbf{g}\cdot\mathbf{\Phi}\right) \mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad + C\int_{S^1}\int_\Omega \left((\varrho^\gamma + \varrho\vartheta + \vartheta^4)\left\{\varrho^{\gamma(a-1)}\right\}_\Omega\right) \mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

The terms on the left-hand side of the inequality have good sign and provide the desired bound of $\varrho^{a\gamma}$, if the right-hand side will be estimated. First,

$$\int_{S^1}\int_\Omega \varrho^\gamma\left\{\varrho^{\gamma(a-1)}\right\}_\Omega \mathrm{d}x\,\mathrm{d}t \leq C\,\|\varrho\|^\gamma_{L^\gamma(L^\gamma)},$$

[7] We denote $\{g\}_\Omega = \frac{1}{|\Omega|}\int_\Omega g\,\mathrm{d}x$.

[8] In fact, $\gamma(a-1) = \frac{\gamma-1}{\gamma+1}$ in 2D, and $\gamma(a-1) = \frac{2\gamma-3}{3}$ in 3D.

which can be put to the left-hand side by means of Young's inequality. From estimate (1.60) we get

$$\begin{aligned}\int_{S^1}\int_{\Omega}\left(\vartheta^4\left\{\varrho^{\gamma(a-1)}\right\}_{\Omega}\right)\mathrm{d}x\,\mathrm{d}t &\le C\int_{S^1}\int_{\Omega}\vartheta^4\,\mathrm{d}x\,\mathrm{d}t\\ &\le C\Big(1+\int_{S^1}\int_{\Omega}\varrho^{\gamma}\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/4}\sup_{t\in S^1}\mathcal{E}^{3/4}(t).\end{aligned}\tag{1.67}$$

Using similar arguments as in (1.62), we gain

$$\int_{S^1}\int_{\Omega}\varrho\vartheta\left\{\varrho^{\gamma(a-1)}\right\}_{\Omega}\mathrm{d}x\,\mathrm{d}t\le C\Big(1+\|\varrho\|^{\alpha\gamma/\gamma-1}_{L^{\gamma+\frac{\gamma-s}{s}}(S^1\times\Omega)}\Big),\tag{1.68}$$

according to (1.64), we are again able to push this term to the left-hand side using Young's inequality. Further,

$$\begin{aligned}\int_{S^1}\int_{\Omega}|(\varrho\mathbf{v}\otimes\mathbf{v}):\nabla\mathbf{\Phi}|\,\mathrm{d}x\,\mathrm{d}t &\le \|\mathbf{v}\|^2_{L^2(L^q)}\|\varrho\nabla\mathbf{\Phi}\|_{L^\infty(L^s)}\\ &\le C\|\varrho\|_{L^\infty(L^\gamma)}\|\nabla\mathbf{\Phi}\|_{L^\infty\left(L^{s\gamma/(\gamma-s)}\right)}\le C\sup_{t\in S^1}\mathcal{E}(t)^{\frac{1}{\gamma}+\frac{\gamma-s}{s\gamma}},\end{aligned}$$

note that exactly this point determines the value of a $\left(\frac{s}{\gamma-s}=\frac{1}{\gamma(a-1)}\right)$, and that $\frac{1}{\gamma}+\frac{\gamma-s}{s\gamma}<1$. For the term with $\frac{\partial\mathbf{\Phi}}{\partial t}$, we will use the renormalized equation of continuity (1.40) with $b(\varrho)=\varrho^{\gamma(a-1)}$ to deduce

$$\begin{aligned}\mathcal{B}\Big[\frac{\partial\varrho^{\gamma(a-1)}}{\partial t}-\Big\{\frac{\partial\varrho^{\gamma(a-1)}}{\partial t}\Big\}_{\Omega}\Big]+\mathcal{B}\left[\operatorname{div}(\varrho^{\gamma(a-1)}\mathbf{v})\right]&\\ +\big(\gamma(a-1)-1\big)\mathcal{B}\Big[\varrho^{\gamma(a-1)}\operatorname{div}\mathbf{v}-\left\{\varrho^{\gamma(a-1)}\operatorname{div}\mathbf{v}\right\}_{\Omega}\Big]&=0;\end{aligned}\tag{1.69}$$

hence we obtain two terms. The first one can be estimated similarly as above, for the other we will distinguish between two and three-dimensional case. In 2D, we have[9]

$$\begin{aligned}&\int_{S^1}\int_{\Omega}\varrho\mathbf{v}\cdot\mathcal{B}\left[\varrho^{\gamma(a-1)}\operatorname{div}\mathbf{v}-\left\{\varrho^{\gamma(a-1)}\operatorname{div}\mathbf{v}\right\}_{\Omega}\right]\mathrm{d}x\,\mathrm{d}t\\ &\quad\le\|\varrho\mathbf{v}\|_{L^2\left(L^{\frac{1}{2-a}}\right)}\left\|\mathcal{B}\left[\varrho^{\gamma(a-1)}\operatorname{div}\mathbf{v}-\left\{\varrho^{\gamma(a-1)}\operatorname{div}\mathbf{v}\right\}_{\Omega}\right]\right\|_{L^2\left(L^{\frac{1}{a-1}}\right)}\\ &\quad\le\|\varrho\|_{L^\infty(L^\gamma)}\|\mathbf{v}\|_{L^2\left(L^{\frac{s}{s-1}}\right)}\left\|\varrho^{\gamma(a-1)}\operatorname{div}\mathbf{v}\right\|_{L^2\left(L^{\frac{2}{2a-1}}\right)},\end{aligned}$$

while in 3D

$$\begin{aligned}&\int_{S^1}\int_{\Omega}\varrho\mathbf{v}\cdot\mathcal{B}\left[\varrho^{\gamma(a-1)}\operatorname{div}\mathbf{v}-\left\{\varrho^{\gamma(a-1)}\operatorname{div}\mathbf{v}\right\}_{\Omega}\right]\mathrm{d}x\,\mathrm{d}t\\ &\quad\le\|\varrho\mathbf{v}\|_{L^2\left(L^{\frac{6\gamma}{6+\gamma}}\right)}\left\|\mathcal{B}\left[\varrho^{\gamma(a-1)}\operatorname{div}\mathbf{v}-\left\{\varrho^{\gamma(a-1)}\operatorname{div}\mathbf{v}\right\}_{\Omega}\right]\right\|_{L^2\left(L^{\frac{6}{6a-5}}\right)}\\ &\quad\le\|\varrho\|_{L^\infty(L^\gamma)}\|\mathbf{v}\|_{L^2(L^6)}\left\|\varrho^{\gamma(a-1)}\operatorname{div}\mathbf{v}\right\|_{L^2\left(L^{\frac{2}{2a-1}}\right)}.\end{aligned}$$

[9] Indeed, $2-a=\frac{1}{\gamma}+\frac{s-1}{s}$ and $\frac{2\cdot2/(2a-1)}{2-2/(2a-1)}=\frac{1}{a-1}$.

Therefore, in both cases

$$\int_{S^1}\int_\Omega \varrho \mathbf{v}\mathcal{B}\left[\varrho^{\gamma(a-1)}\operatorname{div}\mathbf{v} - \left\{\varrho^{\gamma(a-1)}\operatorname{div}\mathbf{v}\right\}_\Omega\right]\mathrm{d}x\,\mathrm{d}t \\ \leq C\sup_{t\in S^1}\mathcal{E}(t)^{\frac{1}{\gamma}}\left\|\varrho^{\gamma(a-1)}\right\|_{L^\infty\left(L^{\frac{1}{a-1}}\right)}\|\operatorname{div}\mathbf{v}\|_{L^2(L^2)} \leq C\sup_{t\in S^1}\mathcal{E}(t)^{\frac{1}{\gamma}+a-1},$$

with $\frac{1}{\gamma}+a-1<1$. Moreover, the embedding $W^{\frac{1}{\gamma(a-1)}}(\Omega)\hookrightarrow L^\infty(\Omega)$, valid for $\gamma<2$, yields

$$\left|\int_{S^1}\int_\Omega \varrho\mathbf{g}\cdot\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t\right| \leq \|\mathbf{g}\|_{L^\infty(L^\infty)}\|\varrho\|_{L^1(L^1)}\|\mathbf{\Phi}\|_{L^\infty(L^\infty)} \leq C.$$

Thus summing up all the estimates together we conclude

$$\int_{S^1}\int_\Omega \varrho^{\gamma+\frac{\gamma-s}{s}}\,\mathrm{d}x\,\mathrm{d}t \leq C\sup_{t\in S^1}\mathcal{E}(t)^\beta,$$

with some $\beta<1$. This estimate can be plugged back into (1.63) in order to get the desired bound

$$\sup_{t\in S^1}\mathcal{E}(t)<\infty.$$

1.2.2 A priori bounds with the radiation on the boundary

Energy estimates

The first observation is again that the conservation of mass (1.46) yields

$$\varrho\in L^\infty\left(S^1;L^1(\Omega)\right). \tag{1.70}$$

Next, by setting $\psi\equiv 1$ in the entropy balance equation (1.42) we obtain

$$\int_{S^1}\int_\Omega\left(\frac{\mathbb{S}(\vartheta,\nabla\mathbf{v}):\nabla\mathbf{v}}{\vartheta}+\frac{\kappa(\vartheta)\left|\nabla\vartheta\right|^2}{\vartheta^2}\right)\mathrm{d}x\,\mathrm{d}t \\ +\int_{S^1}\int_{\partial\Omega}\frac{h(\vartheta)\Theta_0}{\vartheta}\,\mathrm{d}S\,\mathrm{d}t \leq \int_{S^1}\int_{\partial\Omega}h(\vartheta)\,\mathrm{d}S\,\mathrm{d}t.$$

Hence, using the form of h, κ, $\mathbb{S}$, and the Korn inequality (see Theorem 4.4)

$$\|\mathbf{v}\|^2_{L^2(W_0^{1,2}(\Omega))}+\left\|\nabla(\vartheta^{\frac{3}{2}})\right\|^2_{L^2(L^2)}+\|\nabla(\log\vartheta)\|^2_{L^2(L^2)} \\ +\left\|\frac{1}{\vartheta}\right\|_{L^1(S^1\times\partial\Omega)}+\|\vartheta\|^2_{L^2(S^1\times\partial\Omega)} \leq C(1+\|\vartheta\|^3_{L^3(S^1\times\partial\Omega)}). \tag{1.71}$$

Integrating the total energy balance (1.4) over the whole time period yields

$$\int_{S^1}\int_{\partial\Omega}h(\vartheta)(\vartheta-\Theta_0)\,\mathrm{d}S\,\mathrm{d}t = \int_{S^1}\int_\Omega\varrho\mathbf{g}\cdot\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t,$$

id est,

$$\int_{S^1}\int_{\partial\Omega} h_0(\vartheta+\vartheta^4)\,\mathrm{d}S\,\mathrm{d}t \le \left|\int_{S^1}\int_{\Omega}\varrho\mathbf{g}\cdot\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t\right| + C\int_{S^1}\int_{\partial\Omega}(1+\vartheta^3)\Theta_0\,\mathrm{d}S\,\mathrm{d}t,$$
$$\le C\left(1+\|\varrho\|_{L^2(L^r)}\|\mathbf{v}\|_{L^2(L^q)}\right)$$

for $r=\frac{6}{5}$, or any $r>1$, respectively, $\frac{1}{r}+\frac{1}{q}=1$. By estimating the right-hand side of (1.71) by means of this inequality we get in particular

$$\|\vartheta\|^3_{L^3\left(L^{3q/2}\right)} + \|\mathbf{v}\|^2_{L^2(L^q)} \le C\left(1+\|\varrho\|^{6/5}_{L^2(L^r)}\right), \tag{1.72}$$

where we have used the fact that by virtue of the Poincaré inequality

$$\|\vartheta\|^3_{L^3\left(L^{3q/2}\right)} = \left\|\vartheta^{3/2}\right\|^2_{L^2(L^q)} \le C\left\|\vartheta^{3/2}\right\|^2_{L^2(W^{1,2}(\Omega))}$$
$$\le C\left(\|\vartheta\|^3_{L^3(S^1\times\partial\Omega)} + \left\|\nabla(\vartheta^{3/2})\right\|^2_{L^2(L^2)}\right).$$

Note that in contrary to the case without radiation on the boundary, the key estimate (1.72) depends here on a certain norm of ϱ. However, since interpolation with $L^\infty\left(S^1;L^1(\Omega)\right)$ yields

$$\|\varrho\|_{L^2(L^r)} \le C\left(1+\left(\int_{S^1}\left(\int_\Omega\varrho^\gamma\,\mathrm{d}x\right)^{\frac{2(r-1)}{(\gamma-1)r}}\mathrm{d}t\right)^{1/2}\right), \tag{1.73}$$

we obtain

$$\|\vartheta\|_{L^3\left(L^{3q/2}\right)} \le C\left[1+\left(\int_{S^1}\left(\int_\Omega\varrho^\gamma\,\mathrm{d}x\right)^{\frac{2(r-1)}{r(\gamma-1)}}\mathrm{d}t\right)^{\frac{1}{5}}\right], \tag{1.74}$$

$$\|\mathbf{v}\|_{L^2(L^q)} \le C\left[1+\left(\int_{S^1}\left(\int_\Omega\varrho^\gamma\,\mathrm{d}x\right)^{\frac{2(r-1)}{r(\gamma-1)}}\mathrm{d}t\right)^{\frac{3}{10}}\right]. \tag{1.75}$$

From the balance of the total energy (1.4) we can then conclude, as before, with the use of the structural properties of the internal energy that for all $t_1,\ t_2\in S^1$

$$\mathcal{E}(t_1)-\mathcal{E}(t_2) \le C\left(1+\int_{S^1}\mathcal{E}(t)\,\mathrm{d}t\right),$$
$$\sup_{t\in S^1}\mathcal{E}(t) \le C\left(1+\int_{S^1}\mathcal{E}(t)\,\mathrm{d}t\right). \tag{1.76}$$

From the inequalities above we estimate the kinetic energy ($s=\frac{q}{q-2}$)

$$\int_{S^1}\int_\Omega\frac{1}{2}\varrho\,|\mathbf{v}|^2\,\mathrm{d}x\,\mathrm{d}t \le C\,\|\mathbf{v}\|^2_{L^2(L^q)}\,\|\varrho\|_{L^\infty(L^s)}$$
$$\le C\left[1+\left(\int_{S^1}\left(\int_\Omega\varrho^\gamma\,\mathrm{d}x\right)^{\frac{2(r-1)}{r(\gamma-1)}}\mathrm{d}t\right)^{\frac{3}{5}}\|\varrho\|^{\frac{\gamma(s-1)}{s(\gamma-1)}}_{L^\infty(L^\gamma)}\right] \tag{1.77}$$
$$\le C\left(1+\sup_{t\in S^1}E(t)^{\frac{6(r-1)}{5r(\gamma-1)}+\frac{s-1}{s(\gamma-1)}}\right),$$

in 2D since r and s can be chosen close to 1 enough (e.g. $r = s = \min(3, \frac{3}{4-\gamma})$) so that $\frac{6(r-1)}{5r(\gamma-1)} + \frac{s-1}{s(\gamma-1)} < 1$ for $\gamma > 1$, we can absorb the term on the right-hand side using Young's inequality. In the three-dimensional case analogously $\frac{6(r-1)}{5r(\gamma-1)} + \frac{s-1}{s(\gamma-1)} = \frac{1}{5(\gamma-1)} + \frac{1}{3(\gamma-1)} < 1$ for $\gamma > \frac{23}{15}$. Hence we deduce using (1.22) and (1.25)

$$\begin{aligned}\sup_{t\in S^1} E(t) \leq & C\Big(1 + \int_{S^1}\int_\Omega \varrho e(\varrho,\vartheta)\,\mathrm{d}x\,\mathrm{d}t\Big)\\ \leq & C\Big(1 + \int_{S^1}\int_\Omega (\varrho^\gamma + \varrho\vartheta + \vartheta^4)\,\mathrm{d}x\,\mathrm{d}t\Big).\end{aligned} \tag{1.78}$$

The first term on the right-hand side will be left as it is. The last term will be estimated using (1.74) as follows

$$\|\vartheta\|^4_{L^4(L^4)} \leq \|\vartheta\|^3_{L^3\left(L^{3q/2}\right)} \|\vartheta\|_{L^\infty(L^4)} \leq C\left(1 + \|\varrho\|^{\frac{6\gamma(r-1)}{5r(\gamma-1)}}_{L^\gamma(L^\gamma)}\right) \sup_{t\in S^1} E^{1/4}(t), \tag{1.79}$$

where the power of the supremum of the total energy can be put to the left-hand side by means of Young's inequality, since in 2D $\frac{4}{3}\cdot\frac{6\gamma(r-1)}{5(\gamma-1)} < \gamma$ for r sufficiently close to 1 (e.g. $r = \frac{\gamma+1}{2}$), while in 3D the condition $\frac{4}{3}\cdot\frac{\gamma}{5(\gamma-1)} < \gamma$ is satisfied even for every $\gamma > \frac{19}{15}$. Finally, for the remaining term we have in 3D case

$$\begin{aligned}\int_{S^1}\int_\Omega \varrho\vartheta\,\mathrm{d}x\,\mathrm{d}t \leq & \|\varrho\|_{L^{3/2}\left(L^{9/8}\right)} \|\vartheta\|_{L^3(L^9)}\\ \leq & C\left(1 + \|\varrho\|^{\frac{\gamma}{9(\gamma-1)}+\frac{\gamma}{15(\gamma-1)}}_{L^\gamma(L^\gamma)}\right)\\ \leq & C\left(1 + \|\varrho\|^{\frac{8\gamma}{45(\gamma-1)}}_{L^\gamma(L^\gamma)}\right),\end{aligned} \tag{1.80}$$

and similarly in 2D for $p = \frac{3q}{3q-2}$, $p = r = \min(2, \frac{2}{3-\gamma})$

$$\begin{aligned}\int_{S^1}\int_\Omega \varrho\vartheta\,\mathrm{d}x\,\mathrm{d}t \leq & \|\varrho\|_{L^{3/2}(L^p)} \|\vartheta\|_{L^3\left(L^{3q/2}\right)}\\ \leq & C\left(1 + \sup_{t\in S^1} E^{\frac{p-1}{p(\gamma-1)}+\frac{2(r-1)}{5r(\gamma-1)}}(t)\right),\end{aligned} \tag{1.81}$$

thus examining the estimates above we obtain

$$\sup_{t\in S^1} E(t) \leq C\Big(1 + \|\varrho\|^\gamma_{L^\gamma(L^\gamma)}\Big). \tag{1.82}$$

Pressure estimates

It remains to deduce suitable estimates of density, which will be done again by testing the momentum equation with

$$\mathbf{\Phi} = \mathcal{B}\Big[\varrho^{\gamma(a-1)} - \big\{\varrho^{\gamma(a-1)}\big\}_\Omega\Big],$$

where $a > 1$ will be specified later, and $\mathcal{B}$ denotes the Bogovskii operator from Theorem 4.13. Since we assume $\gamma(a-1) \leq 1$; due to the properties of the

Bogovskii operator and (1.70) it follows that $\mathbf{\Phi} \in L^\infty\left(S^1; W^{\frac{1}{\gamma(a-1)}}(\Omega)\right)$, and $\left\{\varrho^{\gamma(a-1)}\right\}_\Omega \in L^\infty(S^1)$, so we obtain using (1.21) and (1.24)

$$\begin{aligned}
&\int_{S^1}\int_\Omega \left(p(\varrho,\vartheta)\varrho^{\gamma(a-1)}\right) \mathrm{d}x\,\mathrm{d}t \\
&\quad\le \int_{S^1}\int_\Omega \left(-\varrho\mathbf{v}\cdot\frac{\partial\mathbf{\Phi}}{\partial t} - (\varrho\mathbf{v}\otimes\mathbf{v}):\nabla\mathbf{\Phi} + \mathbb{S}(\vartheta,\nabla\mathbf{v}):\nabla\mathbf{\Phi} - \varrho\mathbf{g}\cdot\mathbf{\Phi}\right)\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad + C\int_{S^1}\int_\Omega \left((\varrho^\gamma + \varrho\vartheta + \vartheta^4)\left\{\varrho^{\gamma(a-1)}\right\}_\Omega\right)\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

The term on the left-hand side of the inequality is non-negative and it will give us the desired estimate of $\varrho^{a\gamma}$, as soon as the right-hand side will be estimated. Let us begin in the two-dimensional case with the convective term, because it will determine the possible values of a.

$$\begin{aligned}
\int_{S^1}\int_\Omega |(\varrho\mathbf{v}\otimes\mathbf{v}):\nabla\mathbf{\Phi}|\,\mathrm{d}x\,\mathrm{d}t &\le \|\mathbf{v}\|^2_{L^2(L^q)}\|\varrho\nabla\mathbf{\Phi}\|_{L^\infty(L^s)} \\
&\le C\left(1 + \|\varrho\|^{\frac{6\gamma(r-1)}{5r(\gamma-1)}}_{L^\gamma(L^\gamma)}\|\varrho\|_{L^\infty(L^\gamma)}\|\nabla\mathbf{\Phi}\|_{L^\infty\left(L^{\frac{s\gamma}{\gamma-s}}\right)}\right) \\
&\le C\left(1 + \sup_{t\in S^1} E(t)^{\frac{6(r-1)}{5r(\gamma-1)}+\frac{1}{\gamma}+\frac{\gamma-s}{s\gamma}}\right),
\end{aligned}$$

where we have used the properties of the Bogovskii operator, and where we have chosen $\frac{s\gamma}{\gamma-s} = \frac{1}{a-1}$. Note that for p sufficiently close to 1 and s sufficiently close to γ we are able to ensure $\frac{6(r-1)}{5r(\gamma-1)} + \frac{1}{\gamma} + \frac{\gamma-s}{s\gamma} < 1$.[10]

For the term with $\frac{\partial\mathbf{\Phi}}{\partial t}$ we will use the renormalized equation of continuity (1.40) with $b(\varrho) = \varrho^{\gamma(a-1)}$. We obtain two terms, one can be estimated similarly as above, the other as follows

$$\begin{aligned}
&\int_{S^1}\int_\Omega \left|\varrho\mathbf{v}\cdot\mathcal{B}\left[\varrho^{\gamma(a-1)}\operatorname{div}\mathbf{v} - \left\{\varrho^{\gamma(a-1)}\operatorname{div}\mathbf{v}\right\}_\Omega\right]\right|\mathrm{d}x\,\mathrm{d}t \\
&\quad\le C\|\varrho\|_{L^\infty(L^\gamma)}\|\mathbf{v}\|_{L^2\left(L^{\frac{s}{s-1}}\right)}\left\|\varrho^{\gamma(a-1)}\operatorname{div}\mathbf{v}\right\|_{L^2\left(L^{\frac{2}{2a-1}}\right)} \\
&\quad\le C\|\varrho\|_{L^\infty(L^\gamma)}\|\varrho\|^{\frac{3\gamma(p-1)}{5p(\gamma-1)}}_{L^\infty(L^\gamma)}\left\|\varrho^{\gamma(a-1)}\right\|_{L^\infty\left(L^{\frac{1}{a-1}}\right)}\|\operatorname{div}\mathbf{v}\|_{L^2(L^2)} \\
&\quad\le C\left(1 + \sup_{t\in S^1} E(t)^{\frac{1}{\gamma}+a-1+\frac{6(p-1)}{5p(\gamma-1)}}\right),
\end{aligned}$$

where the power on the right-hand side can be again ensured to be less than 1.

While estimating the terms with the temperature, we will use the presence of the radiation on the boundary. Similarly as in (1.79)–(1.81), we have

$$\begin{aligned}
\int_{S^1}\int_\Omega \left(\vartheta^4\left\{\varrho^{\gamma(a-1)}\right\}_\Omega\right)\mathrm{d}x\,\mathrm{d}t &\le C\|\vartheta\|^3_{L^3\left(L^{3q/2}\right)}\|\vartheta\|_{L^\infty(L^4)} \\
&\le C\left(1 + \|\varrho\|^{\frac{6\gamma(r-1)}{5r(\gamma-1)}}_{L^\gamma(L^\gamma)}\right)\sup_{t\in S^1} E^{1/4}(t).
\end{aligned} \tag{1.83}$$

[10] In 3D we will have to proceed later with this term much more carefully in order to obtain the optimal result.

Now, let us turn our attention to the three-dimensional problem. We begin our examination again with the convective term, since it determines the possible values of a and consequently of γ as well.

$$\begin{aligned}\int_{S^1}\int_\Omega |(\varrho\mathbf{v}\otimes\mathbf{v}):\nabla\mathbf{\Phi}|\,\mathrm{d}x\,\mathrm{d}t &\le \|\mathbf{v}\|^2_{L^2(L^6)}\|\varrho\nabla\mathbf{\Phi}\|_{L^\infty\left(L^{3/2}\right)}\\ &\le C\,\|\varrho\|^{\frac{a\gamma}{5(a\gamma-1)}}_{L^{a\gamma}(L^{a\gamma})}\|\varrho\|_{L^\infty(L^p)}\|\nabla\mathbf{\Phi}\|_{L^\infty\left(L^{\frac{p}{\gamma(a-1)}}\right)}\\ &\le \frac{1}{14}\|\varrho\|^{a\gamma}_{L^{a\gamma}(L^{a\gamma})}+C\left(\|\varrho\|_{L^\infty(L^p)}\left\|\varrho^{\gamma(a-1)}\right\|_{L^\infty\left(L^{\frac{p}{\gamma(a-1)}}\right)}\right)^{\frac{5a\gamma-5}{5a\gamma-6}},\end{aligned}\tag{1.84}$$

where we have used an analogy of the estimate (1.75) with $a\gamma$ instead of γ, and the properties of the Bogovskii operator; p satisfies $\frac{2}{3}=\frac{1}{p}+\frac{\gamma(a-1)}{p}$, yielding $p=\frac{3}{2}\big(1+\gamma(a-1)\big)$.

Further, $\|\varrho\|_{L^\infty(L^p)}\left\|\varrho^{\gamma(a-1)}\right\|_{L^\infty\left(L^{\frac{p}{\gamma(a-1)}}\right)}=\|\varrho\|^{1+\gamma(a-1)}_{L^\infty(L^p)}$, and we would like to interpolate as follows

$$\|\varrho\|_{L^\infty(L^p)}\le\|\varrho\|^{\alpha}_{L^\infty(L^\gamma)}\|\varrho\|^{1-\alpha}_{L^\infty(L^1)}.$$

Therefore we need $p\in(1,\gamma)$, which leads to the first constraint on the possible values of a, namely $a<\frac{5\gamma-3}{3\gamma}$. If this condition is satisfied, we get for the interpolation above $\alpha=\dfrac{\gamma}{\gamma-1}\cdot\dfrac{3\gamma a-3\gamma+1}{3\gamma a-3\gamma+3}$. Thus, using (1.70)

$$\begin{aligned}\int_{S^1}\int_\Omega |(\varrho\mathbf{v}\otimes\mathbf{v}):\nabla\mathbf{\Phi}|\,\mathrm{d}x\,\mathrm{d}t \le&\frac{1}{14}\|\varrho\|^{a\gamma}_{L^{a\gamma}(S^1\times\Omega)}+C\left(\|\varrho\|^{\left(1+\gamma(a-1)\right)}_{L^\infty(L^p)}\right)^{\frac{5a\gamma-5}{5a\gamma-6}}\\ \le&\frac{1}{14}\|\varrho\|^{a\gamma}_{L^{a\gamma}(S^1\times\Omega)}+C\left(\|\varrho\|^{\alpha\cdot\left(1+\gamma(a-1)\right)}_{L^\infty(L^\gamma)}\right)^{\frac{5a\gamma-5}{5a\gamma-6}}.\end{aligned}\tag{1.85}$$

The first term can be immediately pushed to the left-hand side, while for the second one, we need

$$\frac{1}{\gamma}\cdot\alpha\cdot\big(1+\gamma(a-1)\big)\cdot\frac{5a\gamma-5}{5a\gamma-6}<1,$$

which leads to a quadratic inequality for the quantity $a\gamma$

$$(5a\gamma-5)\cdot(3a\gamma-3\gamma+1)<3(\gamma-1)\cdot(5a\gamma-6).$$

Denoting $A=a\gamma$, we have

$$15A^2+A(5-30\gamma)+33\gamma-23<0,$$

the discriminant $D_A=5(180\gamma^2-456\gamma+281)$ is definitely positive for all $\gamma>\frac{3}{2}$, so we have to ensure that

$$\frac{30\gamma-5-\sqrt{D_A}}{30}<A<\frac{30\gamma-5+\sqrt{D_A}}{30},\text{ which means}\tag{1.86}$$

$$1<a<1+\frac{-5+\sqrt{D_A}}{30\gamma},\tag{1.87}$$

since we consider only $a > 1$. Therefore we need

$$-5 + \sqrt{D_A} > 0, \qquad \text{id est,}$$
$$180\gamma^2 - 456\gamma + 276 > 0,$$

which yields again the restriction

$$\gamma > \frac{23}{15}. \tag{1.88}$$

Conversely, we are able to choose for each γ satisfying (1.88), $a > 1$ such that we can bound the convective term, namely[11]

$$1 < a < \min\left\{\frac{5\gamma-3}{3\gamma}, 1 + \frac{-5+\sqrt{5\cdot(180\gamma^2-456\gamma+281)}}{30\gamma}, \frac{\gamma+1}{\gamma}\right\}. \tag{1.89}$$

For the term with $\frac{\partial \mathbf{\Phi}}{\partial t}$, we will use the renormalized equation of continuity, see Section 4.6, with $b(\varrho) = \varrho^{\gamma(a-1)}$; we obtain two terms, one can be estimated similarly as above, the other as follows[12]

$$\begin{aligned}
&\int_{S^1}\int_\Omega \left|\varrho\mathbf{v}\cdot\mathcal{B}\left[\varrho^{\gamma(a-1)}\operatorname{div}\mathbf{v} - \left\{\varrho^{\gamma(a-1)}\operatorname{div}\mathbf{v}\right\}_\Omega\right]\right| \mathrm{d}x\,\mathrm{d}t \\
&\quad \le \|\varrho\mathbf{v}\|_{L^2\left(L^{\frac{6p}{p+6}}\right)} \left\|\mathcal{B}\left[\varrho^{\gamma(a-1)}\operatorname{div}\mathbf{v} - \left\{\varrho^{\gamma(a-1)}\operatorname{div}\mathbf{v}\right\}_\Omega\right]\right\|_{L^2\left(L^{\frac{6p}{5p-6}}\right)} \\
&\quad \le C\|\varrho\|_{L^\infty(L^p)}\|\mathbf{v}\|_{L^2(L^6)}\left\|\varrho^{\gamma(a-1)}\operatorname{div}\mathbf{v}\right\|_{L^2\left(L^{\frac{2p}{p+2\gamma(a-1)}}\right)} \\
&\quad \le C\|\varrho\|_{L^\infty(L^p)}\|\mathbf{v}\|_{L^2(L^6)}\left\|\varrho^{\gamma(a-1)}\right\|_{L^\infty\left(L^{\frac{p}{\gamma(a-1)}}\right)}\|\operatorname{div}\mathbf{v}\|_{L^2(L^2)},
\end{aligned}$$

with same p as above. The right-hand side has the same structure as in (1.84), hence we can proceed in the same way as above.

Finally, we conclude

$$\int_{S^1}\int_\Omega \varrho^{a\gamma}\,\mathrm{d}x\,\mathrm{d}t \le C\left(1 + \sup_{t\in S^1} E(t)^\beta\right),$$

with some $\beta < 1$. This estimate can be plugged into (1.82) in order to get the bound

$$\sup_{t\in S^1} E(t) < \infty. \tag{1.90}$$

Moreover, due to the obtained estimates we can derive higher integrability of the temperature on the boundary. Namely, from (1.71) and (1.90) we have

$$\vartheta^{\frac{3}{2}} \in L^2(S^1; W^{1,2}(\Omega)), \qquad \vartheta \in L^\infty\left(S^1; L^4(\Omega)\right)$$

and we can interpolate

$$\begin{aligned}
\|\vartheta\|_{L^{13/3}(\partial\Omega)}^{\frac{13}{3}} &= \int_{\partial\Omega}\vartheta^{\frac{3}{2}\cdot\frac{26}{9}}\,\mathrm{d}S \le C\left\|\vartheta^{\frac{3}{2}}\right\|_{W^{1,2}(\Omega)}\left(\int_\Omega \vartheta^{\frac{3}{2}\cdot 2\cdot(\frac{26}{9}-1)}\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\le C\left\|\vartheta^{\frac{3}{2}}\right\|_{W^{1,2}(\Omega)}\|\vartheta\|_{L^{\frac{17}{3}}(\Omega)}^{\frac{17}{6}} \le C\left\|\vartheta^{\frac{3}{2}}\right\|_{W^{1,2}(\Omega)}\|\vartheta\|_{L^9(\Omega)}^{\frac{3}{2}}\|\vartheta\|_{L^4(\Omega)}^{\frac{4}{3}},
\end{aligned} \tag{1.91}$$

[11] The first quantity is less than the second one for $\gamma > \frac{39}{25}$.

[12] Note $\frac{6p}{7p-6} = \frac{2p}{p+2\gamma(a-1)}$ for $p = \frac{3}{2}\big(1+\gamma(a-1)\big)$.

$$\|\vartheta\|_{L^{13/3}(S^1\times\partial\Omega)}^{13/3} \leq C \left\|\vartheta^{3/2}\right\|_{L^2(W^{1,2}(\Omega))} \|\vartheta\|_{L^3(L^9)}^{3/2} \|\vartheta\|_{L^\infty(L^4)}^{4/3} \leq C. \tag{1.92}$$

Now, we are ready to start the proof of our main theorem in the case of the radiation on the boundary in the three-dimensional setting.

1.3 Approximation

1.3.1 Approximation scheme

Following Feireisl et al. [44], we will approximate the original problem introducing five parameters, namely $N \in \mathbb{N}$ representing the dimension of the finite dimensional space for the velocity field in the Galerkin approximation, $\tau > 0$, and $\zeta > 0$[13] introduced in order to get an information about the time integrability of the velocity, and temperature, respectively, even in the possible vacuum zones, $\varepsilon > 0$ representing the parabolic regularization of the continuity equation, and last, but not least $\delta > 0$ regularizing the pressure and heat flux in order to get higher integrability of the density and the temperature. Moreover, Γ and B denote sufficiently large positive numbers. We will search for $\varrho \geq 0$, $\varrho \in C^\infty(S^1;\, W^{2,p}(\Omega))$, $\mathbf{v}_N$ in some finite dimensional space, and $\vartheta > 0$ such that $\log\vartheta$, $\vartheta \in W^{2,p}(S^1 \times \Omega)$ for any $p < \infty$; we replace the original system by the following approximative version.

We add artificial diffusion and mass production into the continuity equation, and add a corresponding boundary condition in order to conserve the mass, denoting $m = \frac{M_0}{|\Omega|}$

$$\begin{aligned} \frac{\partial\varrho}{\partial t} + \operatorname{div}(\varrho\mathbf{v}_N) - \varepsilon\Delta\varrho + \varepsilon\varrho = \varepsilon m \qquad & \text{in } S^1\times\Omega, \\ \frac{\partial\varrho}{\partial\mathbf{n}} = 0 \qquad & \text{on } S^1\times\partial\Omega. \end{aligned} \tag{1.93}$$

We modify the pressure and consider the Galerkin approximation in the momentum equation. For this purpose we introduce a finite-dimensional subspaces of $L^2(S^1; W_0^{1,2}(\Omega))$ with basis, consisting of $\mathbf{w}^i(t,x) = a^k(t)\mathbf{b}^l(x)$, with $i = i(k,l)$, $i = 1,\dots,N$, which is orthonormal with respect to scalar product $(\mathbf{w}^i, \mathbf{w}^j) = \int_{S^1}\int_\Omega \nabla\mathbf{w}^i : \nabla\mathbf{w}^j \,\mathrm{d}x\,\mathrm{d}t$. Here a^k stands for complete orthonormal basis of goniometric functions, which are smooth and L-periodic,[14] while $\mathbf{b}^l$ forms orthonormal basis of $W_0^{1,2}(\Omega)$, such that all its elements belong to $W^{2,p}(\Omega)$ for any $p < \infty$.

$$\begin{aligned} &\int_{S^1}\int_\Omega \left(\zeta\frac{\partial\mathbf{v}_N}{\partial t}\cdot\mathbf{w}^i + \frac{\partial(\varrho\mathbf{v}_N)}{\partial t}\cdot\mathbf{w}^i - (\varrho\mathbf{v}_N\otimes\mathbf{v}_N) : \nabla\mathbf{w}^i\right)\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{S^1}\int_\Omega \Big(\mathbb{S}(\vartheta,\nabla\mathbf{v}_N) : \nabla\mathbf{w}^i - \big(p(\varrho,\vartheta) + \delta(\varrho^\Gamma + \varrho^2)\big)\operatorname{div}\mathbf{w}^i\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_{S^1}\int_\Omega \left(-\varepsilon\nabla\varrho\cdot\nabla\mathbf{v}_N\mathbf{w}^i + \frac{1}{2}\varepsilon(m-\varrho)\mathbf{v}_N\cdot\mathbf{w}^i + \varrho\mathbf{g}\cdot\mathbf{w}^i\right)\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{1.94}$$

[13] We will finally set $\zeta = \delta$, but we keep this notation for the purpose of clarity.

[14] We can take for example $\cos\left(\frac{k\pi t}{L}\right)$, and $\sin\left(\frac{(k+1)\pi t}{L}\right)$ for k odd, or even, respectively.

We transform the regularized internal energy equation, which we see as an equation for the temperature, by means of the so-called Kirchhoff transform, see e.g. Roubíček [133, Example 2.74]

$$\Phi(g) = \int_0^g \big(\kappa(\mathrm{e}^z)\mathrm{e}^z + \delta \mathrm{e}^{(B+1)z} + \delta\big)\mathrm{d}z. \tag{1.95}$$

Note that, since the integrand of the integral is measurable and greater than δ, Φ is continuous, increasing, and one-to-one with Φ^{-1} Lipschitz continuous, in particular having a linear growth. We get

$$\begin{aligned}
-\tau\frac{\partial^2\Phi(\log\vartheta)}{\partial t^2} + \zeta\frac{\partial\vartheta}{\partial t} + \tau\Phi(\log\vartheta) + \frac{\partial(\varrho e)}{\partial t} - \operatorname{div}\nabla\Phi(\log\vartheta) + \operatorname{div}(\varrho e\mathbf{v}_N)&\\
= \mathbb{S}(\vartheta,\nabla\mathbf{v}_N) : \nabla\mathbf{v}_N - p(\varrho,\vartheta)\operatorname{div}\mathbf{v}_N + \varepsilon\delta(\Gamma\varrho^{\Gamma-2}+2)\left|\nabla\varrho\right|^2 + \delta\vartheta^{-1}&\\
\text{in } S^1\times\Omega,&\\
\big(\kappa(\vartheta) + \delta\vartheta^B + \delta\vartheta^{-1}\big)\frac{\partial\vartheta}{\partial\mathbf{n}} = h(x,\vartheta)(\Theta_0 - \vartheta) \qquad \text{on } S^1\times\partial\Omega.&
\end{aligned} \tag{1.96}$$

Since we have in our definition of the solution the entropy equation instead of the energy equation we will present now also its approximative version. It can be derived by dividing (1.96) by temperature ϑ with use of (1.93)

$$\begin{aligned}
&-\tau\partial_t\left(\frac{\Phi'(\log\vartheta)}{\vartheta}\partial_t(\log\vartheta)\right) - \tau\frac{\Phi'(\log\vartheta)}{\vartheta^3}\left(\frac{\partial\vartheta}{\partial t}\right)^2 + \zeta\frac{\partial\log\vartheta}{\partial t} + \frac{\partial(\varrho s)}{\partial t}\\
&\quad + \tau\frac{\Phi(\log\vartheta)}{\vartheta} + \big(\operatorname{div}(\varrho\mathbf{v}_N) + \partial_t\varrho\big)\frac{\varrho e + p - \varrho\vartheta s}{\varrho\vartheta} + \operatorname{div}(\varrho s\mathbf{v}_N)\\
&\quad - \operatorname{div}\Big(\big(\kappa(\vartheta) + \delta\vartheta^B + \delta\vartheta^{-1}\big)\frac{\nabla\vartheta}{\vartheta}\Big) = \frac{1}{\vartheta}\mathbb{S}(\vartheta,\nabla\mathbf{v}_N) : \nabla\mathbf{v}_N\\
&\quad + \big(\kappa(\vartheta) + \delta\vartheta^B + \delta\vartheta^{-1}\big)\frac{\left|\nabla\vartheta\right|^2}{\vartheta^2} + \delta\frac{1}{\vartheta^2} + \frac{\varepsilon\delta}{\vartheta}(\Gamma\varrho^{\Gamma-2}+2)\left|\nabla\varrho\right|^2\\
&\qquad \text{in } S^1\times\Omega.
\end{aligned} \tag{1.97}$$

1.3.2 Existence of approximate solutions for fixed parameters

The main aim of this subsection is to show the following existence result for the approximative problem.

Lemma 1.6. *For an arbitrary fixed set of parameters $N\in\mathbb{N}$, τ, ζ, ε, $\delta>0$ such that $\varepsilon\ll\delta$, there exists at least one solution to the approximate scheme, id est $\varrho\geq 0$, $\varrho\in C^\infty(S^1, W^{2,p}(\Omega))$, $\mathbf{v}_N\in Lin\{\mathbf{w}^i\}_{i=1}^N$, and $\vartheta>0$, with $\log\vartheta$, $\vartheta\in W^{2,p}(S^1\times\Omega)$, such that (1.93), (1.94), and (1.96) hold.*

First of all, we observe that as soon as we have the velocity field, we are able to recover the density, namely

Proposition 1.7. *For any velocity field $\tilde{\mathbf{v}}_N\in Lin\{\mathbf{w}^i\}_{i=1}^N$, there exists a density $\varrho\in C^\infty(S^1; W^{2,p}(\Omega))$ satisfying $\varrho\geq 0$,*

$$\begin{aligned}
\partial_t\varrho + \operatorname{div}(\varrho\tilde{\mathbf{v}}_N) - \varepsilon\Delta\varrho + \varepsilon\varrho = \varepsilon m \ \text{ in } S^1\times\Omega&\\
\frac{\partial\varrho}{\partial\mathbf{n}} = 0 \ \text{ on } S^1\times\partial\Omega.&
\end{aligned} \tag{1.98}$$

Moreover,

$$\int_\Omega \varrho \, \mathrm{d}x = m|\Omega| = M_0. \tag{1.99}$$

Proof of Proposition 1.7. The proof is standard and it is based on application of fixed point argument and the regularity properties of parabolic equations. First, we fix $\tilde{\varrho}$ and find the solution ϱ to problem

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\tilde{\varrho}\tilde{\mathbf{v}}_N) - \varepsilon\Delta\varrho + \varepsilon\varrho = \varepsilon m \text{ in } S^1 \times \Omega, \\ \frac{\partial \varrho}{\partial \mathbf{n}} = 0 \text{ on } S^1 \times \partial\Omega. \end{aligned} \tag{1.100}$$

via the Galerkin method.[15] Next, we consider the mapping $\tilde{\varrho} \mapsto \varrho$ and find its fixed point in $W^{1,p}(S^1 \times \Omega)$ by means of Theorem 4.9. Note that this is the moment where we use the fact that we assume quite smooth ($C^{2+\eta}$) boundary of the domain. Relation (1.99) is a direct consequence of (1.98) integrated over Ω and the uniqueness argument for ordinary differential equations. Finally, we have to ensure that the density is non-negative. For this purpose we multiply the equation by a characteristic function of the set $\{\varrho < 0\}$. Assuming that $\{\varrho = 0\}$ is manifold regular enough, we obtain $|\{\varrho < 0\}| = 0$ in a straightforward way. Otherwise, we have to regularize the set by considering $\{\varrho = \varepsilon_n\}$ instead of $\{\varrho = 0\}$ and then pass to the limit with $\varepsilon_n \to 0^+$. □

We will apply Theorem 4.9 on the mapping

$$\mathcal{T} : Lin\left\{\mathbf{w}^i\right\}_{i=1}^N \times W^{1,p}(S^1 \times \Omega) \mapsto Lin\left\{\mathbf{w}^i\right\}_{i=1}^N \times W^{1,p}(S^1 \times \Omega),$$

$\mathcal{T}$ is defined as a solving operator $\big(\mathcal{T}(\tilde{\mathbf{v}}_N, \log\tilde{\vartheta}) = (\mathbf{v}_N, \log\vartheta)\big)$ to the following linearized system

$$\begin{aligned} &\int_{S^1}\int_\Omega \left(\zeta\frac{\partial \mathbf{v}_N}{\partial t}\cdot\mathbf{w}^i + \frac{\partial(\varrho\tilde{\mathbf{v}}_N)}{\partial t}\cdot\mathbf{w}^i - (\varrho\tilde{\mathbf{v}}_N \otimes \tilde{\mathbf{v}}_N) : \nabla\mathbf{w}^i\right)\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{S^1}\int_\Omega \left(\mathbb{S}(\tilde{\vartheta}, \nabla\mathbf{v}_N) : \nabla\mathbf{w}^i - \left(p(\varrho,\tilde{\vartheta}) + \delta(\varrho^\Gamma + \varrho^2)\right)\operatorname{div}\mathbf{w}^i\right)\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_{S^1}\int_\Omega \left(-\varepsilon\big(\nabla\varrho\cdot\nabla\tilde{\mathbf{v}}_N\big)\cdot\mathbf{w}^i + \frac{1}{2}\varepsilon(m-\varrho)\tilde{\mathbf{v}}_N\cdot\mathbf{w}^i + \varrho\mathbf{g}\cdot\mathbf{w}^i\right)\mathrm{d}x\,\mathrm{d}t \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1,\dots,N \end{aligned} \tag{1.101}$$

$$\begin{aligned} &-\tau\frac{\partial^2\Phi(\log\vartheta)}{\partial t^2} + \zeta\frac{\partial\tilde{\vartheta}}{\partial t} + \tau\Phi(\log\vartheta) + \frac{\partial(\varrho\tilde{e})}{\partial t} - \operatorname{div}\nabla\Phi(\log\vartheta) + \operatorname{div}(\varrho\tilde{e}\tilde{\mathbf{v}}_N) \\ &\quad = \mathbb{S}(\tilde{\vartheta}, \nabla\tilde{\mathbf{v}}_N) : \nabla\tilde{\mathbf{v}}_N - p(\varrho,\tilde{\vartheta})\operatorname{div}\tilde{\mathbf{v}}_N + \varepsilon\delta(\Gamma\varrho^{\Gamma-2} + 2)\,|\nabla\varrho|^2 + \delta\tilde{\vartheta}^{-1} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } S^1\times\Omega, \\ &\qquad\qquad \frac{\partial\Phi(\log\vartheta)}{\partial\mathbf{n}} = h(x,\tilde{\vartheta})(\Theta_0 - \tilde{\vartheta}), \qquad \text{on } S^1\times\partial\Omega, \end{aligned} \tag{1.102}$$

[15] There exists a unique solution to the corresponding evolutionary problem for a fixed initial condition, and the periodicity map has a fixed point according to the Schauder fixed point theorem 4.8, see Feireisl et al. [43, Proposition 2.1] for details.

where ϱ is defined as a solution to (1.98) from Proposition 1.7, Φ is as above. We have introduced the notation $\tilde{e} = e(\varrho, \tilde{\vartheta})$.

Concerning the momentum equation, it is easy to show the existence of solution to the corresponding system of linear algebraic equations, using Korn's inequality from Theorem 4.4 and the Brouwer fixed point theorem 4.7. Note especially, that we consider the Galerkin approximation in time as well as in space.

Proposition 1.8. *For any $\tilde{\mathbf{v}}_N \in Lin\,\{\mathbf{w}^i\}_{i=1}^N$, $\tilde{\vartheta} \in W^{1,p}(S^1 \times \Omega)$, and a corresponding $\varrho \in W^{1,p}(S^1 \times \Omega)$ from Proposition 1.7, there exists a unique solution to* (1.101)*. Furthermore, it satisfies $\mathbf{v}_N \in Lin\,\{\mathbf{w}^i\}_{i=1}^N$.*

The second part of the solving operator $\mathcal{T}$ is defined through the energy equation.

Proposition 1.9. *For any $\tilde{\mathbf{v}}_N \in Lin\,\{\mathbf{w}^i\}_{i=1}^N$, $\tilde{\vartheta} \in W^{1,p}(S^1 \times \Omega)$, and a corresponding ϱ from Proposition 1.7, there exists a uniquely defined $\vartheta > 0$ such that ϑ and $\log\vartheta \in W^{2,p}(S^1 \times \Omega)$, satisfying (1.102).*

Proof of Proposition 1.9. The central point in the proof is that instead of searching directly for ϑ, we solve the system for $\log\vartheta$, and then set $\vartheta := \mathrm{e}^{\log\vartheta}$, which immediately implies $\vartheta > 0$. More precisely, we solve the $(d+1)$-dimensional elliptic problem for Z, see Theorem 4.11 and the remark below

$$\begin{aligned}
&-\tau\frac{\partial^2 Z}{\partial t^2} + \zeta\frac{\partial\tilde{\vartheta}}{\partial t} + \tau Z + \frac{\partial(\varrho\tilde{e})}{\partial t} - \operatorname{div}\nabla Z + \operatorname{div}(\varrho\tilde{e}\tilde{\mathbf{v}}_N) - \delta\tilde{\vartheta}^{-1} \\
&\qquad = \mathbb{S}(\tilde{\vartheta}, \nabla\tilde{\mathbf{v}}_N) : \nabla\tilde{\mathbf{v}}_N - p(\varrho, \tilde{\vartheta})\operatorname{div}\tilde{\mathbf{v}}_N + \varepsilon\delta(\Gamma\varrho^{\Gamma-2} + 2)\,|\nabla\varrho|^2 \ \text{ in } S^1 \times \Omega, \\
&\qquad\qquad\qquad \frac{\partial Z}{\partial\mathbf{n}} = h(x, \tilde{\vartheta})(\Theta_0 - \tilde{\vartheta}) \qquad \text{on } S^1 \times \partial\Omega,
\end{aligned} \tag{1.103}$$

and then define $\log\vartheta = \Phi^{-1}(Z)$, which is well-defined thanks to (1.95) and the note below it. □

To summarize, above defined operator $\mathcal{T}$ is a compact and continuous operator from $Lin\,\{\mathbf{w}^i\}_{i=1}^N \times W^{1,p}(S^1 \times \Omega)$ into itself. Thus, in order to apply Theorem 4.9, it remains to verify the boundedness of the possible fixed points

$$\ell\mathcal{T}(\mathbf{v}_N, \vartheta) = (\mathbf{v}_N, \vartheta), \text{ for } 0 \le \ell \le 1 \tag{1.104}$$

in the space $Lin\,\{\mathbf{w}^i\}_{i=1}^N \times W^{1,p}(S^1 \times \Omega)$.

Formula (1.104) is nothing but

$$\begin{aligned}
&\int_{S^1}\int_\Omega \left(\zeta\frac{\partial\mathbf{v}_N}{\partial t}\cdot\mathbf{w}^i + \ell\frac{\partial(\varrho\mathbf{v}_N)}{\partial t}\cdot\mathbf{w}^i - \ell(\varrho\mathbf{v}_N\otimes\mathbf{v}_N) : \nabla\mathbf{w}^i\right)\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{S^1}\int_\Omega \left(\mathbb{S}(\vartheta, \nabla\mathbf{v}_N) : \nabla\mathbf{w}^i - \ell\left(p(\varrho,\vartheta) + \delta(\varrho^\Gamma + \varrho^2)\right)\operatorname{div}\mathbf{w}^i\right)\mathrm{d}x\,\mathrm{d}t \\
&\quad = \ell\int_{S^1}\int_\Omega \left(-\varepsilon\nabla\varrho\cdot\nabla\mathbf{v}_N\mathbf{w}^i + \frac{1}{2}\varepsilon(m-\varrho)\mathbf{v}_N\cdot\mathbf{w}^i + \varrho\mathbf{g}\cdot\mathbf{w}^i\right)\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{1.105}$$

$$
\begin{aligned}
-\tau\frac{\partial^2\Phi(\log\vartheta)}{\partial t^2} + \ell\zeta\frac{\partial\vartheta}{\partial t} + \tau\Phi(\log\vartheta) + \ell\frac{\partial(\varrho e)}{\partial t} - \operatorname{div}\nabla\Phi(\log\vartheta) + \ell\operatorname{div}(\varrho e\mathbf{v}_N) \\
= \ell\mathbb{S}(\vartheta,\nabla\mathbf{v}_N):\nabla\mathbf{v}_N - \ell p(\varrho,\vartheta)\operatorname{div}\mathbf{v}_N + \ell\varepsilon\delta(\Gamma\varrho^{\Gamma-2}+2)\left|\nabla\varrho\right|^2 + \ell\delta\vartheta^{-1} \\
\text{in } S^1\times\Omega, \\
\left(\kappa(\vartheta)+\delta\vartheta^B+\delta\vartheta^{-1}\right)\frac{\partial\vartheta}{\partial\mathbf{n}} = \ell h(x,\vartheta)(\Theta_0-\vartheta) \qquad \text{on } S^1\times\partial\Omega,
\end{aligned}
\tag{1.106}
$$

where ϱ satisfies (1.93), and Φ is given by (1.95). Using $\mathbf{v}_N$ as a test function in momentum equation (1.105) with help of integration by parts yields

$$
\begin{aligned}
&\int_{S^1}\int_\Omega\left(\zeta\frac{\partial\mathbf{v}_N}{\partial t}\cdot\mathbf{v}_N + \ell\frac{\partial(\varrho\mathbf{v}_N)}{\partial t}\cdot\mathbf{v}_N - \ell(\varrho\mathbf{v}_N\otimes\mathbf{v}_N):\nabla\mathbf{v}_N\right)\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{S^1}\int_\Omega\left(\mathbb{S}(\vartheta,\nabla\mathbf{v}_N):\nabla\mathbf{v}_N - \ell\left(p(\varrho,\vartheta)+\delta(\varrho^\Gamma+\varrho^2)\right)\operatorname{div}\mathbf{v}_N\right)\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \ell\int_{S^1}\int_\Omega\left(\frac{1}{2}\varepsilon\Delta\varrho\left|\mathbf{v}_N\right|^2 + \frac{1}{2}\varepsilon(m-\varrho)\left|\mathbf{v}_N\right|^2 + \varrho\mathbf{g}\cdot\mathbf{v}_N\right)\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{1.107}
$$

Using (1.93) multiplied on $\ell\frac{1}{2}\left|\mathbf{v}_N\right|^2$ with another integration by parts[16] we get

$$
\begin{aligned}
&\int_{S^1}\int_\Omega\left(\mathbb{S}(\vartheta,\nabla\mathbf{v}_N):\nabla\mathbf{v}_N\right)\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \int_{S^1}\int_\Omega\left(\ell\left(p(\varrho,\vartheta)+\delta(\varrho^\Gamma+\varrho^2)\right)\operatorname{div}\mathbf{v}_N + \varrho\mathbf{g}\cdot\mathbf{v}_N\right)\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{1.108}
$$

Integrating the energy equation (1.106) over $S^1\times\Omega$ we obtain with use of the boundary condition

$$
\begin{aligned}
&\tau\int_{S^1}\int_\Omega\Phi(\log\vartheta)\,\mathrm{d}x\,\mathrm{d}t + \ell\int_{S^1}\int_{\partial\Omega}h(x,\vartheta)\vartheta\,\mathrm{d}S\,\mathrm{d}t \\
&\qquad = \ell\int_{S^1}\int_\Omega\left(\mathbb{S}(\vartheta,\nabla\mathbf{v}_N):\nabla\mathbf{v}_N - p(\varrho,\vartheta)\operatorname{div}\mathbf{v}_N + \delta\vartheta^{-1}\right)\mathrm{d}x\,\mathrm{d}t \\
&\quad + \ell\varepsilon\delta\int_{S^1}\int_\Omega(\Gamma\varrho^{\Gamma-2}+2)\left|\nabla\varrho\right|^2\mathrm{d}x\,\mathrm{d}t + \ell\int_{S^1}\int_{\partial\Omega}h(x,\vartheta)\Theta_0\,\mathrm{d}S\,\mathrm{d}t.
\end{aligned}
\tag{1.109}
$$

Further, we get renormalized version of the continuity equation by multliplying (1.93) by $\frac{\beta}{\beta-1}\varrho^{\beta-1}$ after some obvious manipulations

$$
\begin{aligned}
&\varepsilon\beta\int_{S^1}\int_\Omega\left(\frac{1}{\beta-1}\varrho^\beta + \varrho^{\beta-2}\left|\nabla\varrho\right|^2\right)\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_{S^1}\int_\Omega\varrho^\beta\operatorname{div}\mathbf{v}_N\,\mathrm{d}x\,\mathrm{d}t = \varepsilon\frac{\beta}{\beta-1}\int_{S^1}\int_\Omega m\varrho^{\beta-1}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{1.110}
$$

[16]The terms in the form of time derivative vanish due to the time-periodic condition.

In order to get the total energy balance, we sum up (1.108), (1.109) and (1.110) with $\beta = 2, \Gamma$ multiplied by $\delta\ell$. This reads

$$\begin{aligned}
(1-\ell)\int_{S^1}\int_\Omega \big(\mathbb{S}(\vartheta,\nabla\mathbf{v}_N):\nabla\mathbf{v}_N\big)\,\mathrm{d}x\,\mathrm{d}t + \tau\int_{S^1}\int_\Omega \Phi(\log\vartheta)\,\mathrm{d}x\,\mathrm{d}t\\
+\ell\int_{S^1}\int_{\partial\Omega} h(x,\vartheta)\vartheta\,\mathrm{d}S\,\mathrm{d}t + \varepsilon\delta\ell\int_{S^1}\int_\Omega\Big(\frac{\Gamma}{\Gamma-1}\varrho^\Gamma + 2\varrho^2\Big)\,\mathrm{d}x\,\mathrm{d}t\\
=\ell\int_{S^1}\int_\Omega\Big(\varrho\mathbf{g}\cdot\mathbf{v}_N + \delta\vartheta^{-1}\Big)\,\mathrm{d}x\,\mathrm{d}t + \ell\int_{S^1}\int_{\partial\Omega} h(x,\vartheta)\Theta_0\,\mathrm{d}S\,\mathrm{d}t\\
+\ell\int_{S^1}\int_\Omega\Big(\varepsilon\delta\frac{\Gamma}{\Gamma-1}m\varrho^{\Gamma-1} + 2\varepsilon\delta m\varrho\Big)\,\mathrm{d}x\,\mathrm{d}t. \quad (1.111)
\end{aligned}$$

The last two integrals on the right-hand side of (1.111) can be pushed to the left-hand side by means of Young's inequality yielding

$$\begin{aligned}
(1-\ell)\int_{S^1}\int_\Omega \big(\mathbb{S}(\vartheta,\nabla\mathbf{v}_N):\nabla\mathbf{v}_N\big)\,\mathrm{d}x\,\mathrm{d}t + \tau\int_{S^1}\int_\Omega \Phi(\log\vartheta)\,\mathrm{d}x\,\mathrm{d}t\\
+\ell\int_{S^1}\int_{\partial\Omega} h(x,\vartheta)\vartheta\,\mathrm{d}S\,\mathrm{d}t + \varepsilon\delta\ell\int_{S^1}\int_\Omega\Big(\frac{\Gamma}{\Gamma-1}\varrho^\Gamma + 2\varrho^2\Big)\,\mathrm{d}x\,\mathrm{d}t\\
\le C\Big(1+\ell\int_{S^1}\int_\Omega\big(\varrho\mathbf{g}\cdot\mathbf{v}_N + \delta\vartheta^{-1}\big)\,\mathrm{d}x\,\mathrm{d}t\Big). \quad (1.112)
\end{aligned}$$

By similar arguments which lead from (1.96) to (1.97) we can obtain from (1.106) the following entropy identity

$$\begin{aligned}
-\tau\partial_t\Big(\frac{\Phi'(\log\vartheta)}{\vartheta}\partial_t(\log\vartheta)\Big) - \tau\frac{\Phi'(\log\vartheta)}{\vartheta^3}\Big(\frac{\partial\vartheta}{\partial t}\Big)^2 + \ell\zeta\frac{\partial\log\vartheta}{\partial t} + \ell\frac{\partial(\varrho s)}{\partial t}\\
+\tau\frac{\Phi(\log\vartheta)}{\vartheta} + \ell\big(\operatorname{div}(\varrho\mathbf{v}_N)+\partial_t\varrho\big)\frac{\varrho e + p - \varrho\vartheta s}{\varrho\vartheta} + \operatorname{div}(\varrho s\mathbf{v}_N)\\
-\operatorname{div}\Big(\big(\kappa(\vartheta)+\delta\vartheta^B+\delta\vartheta^{-1}\big)\frac{\nabla\vartheta}{\vartheta}\Big) = \ell\frac{1}{\vartheta}\mathbb{S}(\vartheta,\nabla\mathbf{v}_N):\nabla\mathbf{v}_N\\
+\big(\kappa(\vartheta)+\delta\vartheta^B+\delta\vartheta^{-1}\big)\frac{|\nabla\vartheta|^2}{\vartheta^2} + \ell\delta\frac{1}{\vartheta^2} + \ell\frac{\varepsilon\delta}{\vartheta}(\Gamma\varrho^{\Gamma-2}+2)\,|\nabla\varrho|^2\\
\text{in } S^1\times\Omega, \quad (1.113)
\end{aligned}$$

which can be integrated over $S^1\times\Omega$ to get

$$\begin{aligned}
\int_{S^1}\int_\Omega\big(\kappa(\vartheta)+\delta\vartheta^B+\delta\vartheta^{-1}\big)\frac{|\nabla\vartheta|^2}{\vartheta^2}\,\mathrm{d}x\,\mathrm{d}t + \tau\int_{S^1}\int_\Omega\Phi'(\log\vartheta)\frac{(\partial_t\vartheta)^2}{\vartheta^3}\,\mathrm{d}x\,\mathrm{d}t\\
+\ell\int_{S^1}\int_{\partial\Omega}\frac{1}{\vartheta}h(x,\vartheta)\Theta_0\,\mathrm{d}S\,\mathrm{d}t + \ell\int_{S^1}\int_\Omega\Big(\frac{1}{\vartheta}\mathbb{S}(\vartheta,\nabla\mathbf{v}_N):\nabla\mathbf{v}_N + \frac{\delta}{\vartheta^2}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
+\varepsilon\delta\ell\int_{S^1}\int_\Omega\frac{1}{\vartheta}(\Gamma\varrho^{\Gamma-2}+2)\,|\nabla\varrho|^2\,\mathrm{d}x\,\mathrm{d}t = \ell\int_{S^1}\int_{\partial\Omega}h(x,\vartheta)\Theta_0\,\mathrm{d}S\,\mathrm{d}t\\
+\tau\int_{S^1}\int_\Omega\frac{\Phi(\log\vartheta)}{\vartheta}\,\mathrm{d}x\,\mathrm{d}t + \int_{S^1}\int_\Omega\big(\operatorname{div}(\varrho\mathbf{v}_N)+\partial_t\varrho\big)\frac{\varrho e+p-\varrho\vartheta s}{\varrho\vartheta}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

In order to estimate the last term on the right-hand side, we will use the continuity equation (1.93). Hence the last integral transforms into

$$\varepsilon\int_{S^1}\int_\Omega\big(m-\varrho+\Delta\varrho\big)\frac{\varrho e+p-\varrho\vartheta s}{\varrho\vartheta}\,\mathrm{d}x\,\mathrm{d}t. \quad (1.114)$$

The terms coming from (1.114) can be treated similarly as in [115]. Namely, according to (1.24)–(1.25) the term

$$\varepsilon \int_{S^1} \int_\Omega \varrho \Big(\frac{e(\varrho,\vartheta)}{\vartheta} + \frac{p(\varrho,\vartheta)}{\varrho\vartheta} \Big) \,\mathrm{d}x\,\mathrm{d}t \geq \varepsilon c \int_{S^1} \int_\Omega \frac{\varrho^\gamma}{\vartheta} \,\mathrm{d}x\,\mathrm{d}t$$

has negative sign on the right-hand side, while from (1.29) we have

$$\begin{aligned} \varepsilon \int_{S^1} \int_\Omega \varrho s(\varrho,\vartheta) \,\mathrm{d}x\,\mathrm{d}t &\leq \varepsilon \int_{S^1} \int_\Omega C\varrho (1 + |\log \varrho| + |\log \vartheta| + \vartheta^3) \,\mathrm{d}x\,\mathrm{d}t \\ &\leq \varepsilon \Big(C + C \int_{S^1} \int_\Omega \varrho^\gamma \,\mathrm{d}x\,\mathrm{d}t + \frac{c}{4} \int_{S^1} \int_\Omega \frac{\varrho^\gamma}{\vartheta} \,\mathrm{d}x\,\mathrm{d}t + C \|\vartheta\|^3_{L^{3B}(S^1\times\Omega)} \Big). \end{aligned} \tag{1.115}$$

Furthermore,

$$\begin{aligned} \varepsilon m \int_{S^1} \int_\Omega \frac{\varrho e(\varrho,\vartheta) + p(\varrho,\vartheta)}{\varrho\vartheta} \,\mathrm{d}x\,\mathrm{d}t &\leq C\varepsilon m \int_{S^1} \int_\Omega 1 + \frac{\varrho^{\gamma-1}}{\vartheta} + \vartheta^4 \,\mathrm{d}x\,\mathrm{d}t \\ &\leq \varepsilon \Big(C + \frac{c}{4} \int_{S^1} \int_\Omega \frac{\varrho^\gamma}{\vartheta} \,\mathrm{d}x\,\mathrm{d}t + C \int_{S^1} \int_\Omega \frac{1}{\vartheta} \,\mathrm{d}x\,\mathrm{d}t + C \|\vartheta\|^4_{L^{3B}(S^1\times\Omega)} \Big). \end{aligned} \tag{1.116}$$

Similarly, by virtue of Gibbs' relation (1.12) we can recognize the terms with the negative sign

$$\begin{aligned} \varepsilon \int_{S^1} \int_\Omega & \Delta\varrho \frac{\varrho e(\varrho,\vartheta) + p(\varrho,\vartheta) - \varrho\vartheta s(\varrho,\vartheta)}{\varrho\vartheta} \,\mathrm{d}x\,\mathrm{d}t \\ &= -\varepsilon \int_{S^1} \int_\Omega |\nabla\varrho|^2 \frac{\partial}{\partial\varrho} \Big(\frac{e(\varrho,\vartheta)}{\vartheta} + \frac{p(\varrho,\vartheta)}{\varrho\vartheta} - s(\varrho,\vartheta) \Big) \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad - \varepsilon \int_{S^1} \int_\Omega \nabla\varrho \cdot \nabla\vartheta \frac{\partial}{\partial\vartheta} \Big(\frac{e(\varrho,\vartheta)}{\vartheta} + \frac{p(\varrho,\vartheta)}{\varrho\vartheta} - s(\varrho,\vartheta) \Big) \,\mathrm{d}x\,\mathrm{d}t \\ &= -\varepsilon \int_{S^1} \int_\Omega |\nabla\varrho|^2 \frac{1}{\varrho\vartheta} \frac{\partial p(\varrho,\vartheta)}{\partial\varrho} \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \varepsilon \int_{S^1} \int_\Omega \nabla\varrho \cdot \nabla\vartheta \frac{1}{\vartheta^2} \Big(e(\varrho,\vartheta) + \varrho \frac{\partial e(\varrho,\vartheta)}{\partial\varrho} \Big) \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{1.117}$$

Due to (1.27), we can put the first term to the left-hand side, while the other one can be estimated using Young's inequality as follows

$$\begin{aligned} \varepsilon \int_{S^1} \int_\Omega & \nabla\varrho \cdot \nabla\vartheta \frac{1}{\vartheta^2} C \Big(\varrho^{\gamma-1} + \vartheta \Big) \,\mathrm{d}x\,\mathrm{d}t \\ &\leq \varepsilon C \int_{S^1} \int_\Omega \frac{\varrho^{\gamma-1}\nabla\varrho}{\sqrt{\vartheta}} \cdot \frac{\nabla\vartheta}{\vartheta^{3/2}} \,\mathrm{d}x\,\mathrm{d}t + \varepsilon C \int_{S^1} \int_\Omega \Big(\frac{\nabla\varrho \cdot \nabla\vartheta}{\vartheta} \Big) \,\mathrm{d}x\,\mathrm{d}t \\ &\leq \frac{\varepsilon\delta}{4} \int_{S^1} \int_\Omega \frac{1}{\vartheta} \Big(1 + \varrho^{\Gamma-2} \Big) |\nabla\varrho|^2 \,\mathrm{d}x\,\mathrm{d}t + C(\delta)\varepsilon \int_{S^1} \int_\Omega \Big(\frac{|\nabla\vartheta|^2}{\vartheta^3} + \frac{|\nabla\vartheta|^2}{\vartheta} \Big) \,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{1.118}$$

provided $\Gamma \geq 2\gamma$. We will choose $\varepsilon \ll \delta$ in such a way that $C(\delta)\varepsilon < \frac{\delta}{2}$, so both terms can be pushed to the left-hand side. The last remaining term coming from integral (1.114) needs a special treatment

$$
\begin{aligned}
\varepsilon m & \iint_{S^1\times\Omega} s(\varrho,\vartheta)\,\mathrm{d}x\,\mathrm{d}t \\
&\geq \varepsilon m\Big(\iint_{\{\varrho<1,\vartheta>1\}} s_0(\varrho,\vartheta)\,\mathrm{d}x\,\mathrm{d}t + \iint_{\{\varrho<1,\vartheta<1\}} \big(s_0(\varrho,\vartheta)-c\log\vartheta\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad + \iint_{\{\varrho<1,\vartheta<1\}} c\log\vartheta\,\mathrm{d}x\,\mathrm{d}t + \iint_{\{\varrho>1\}} s_0(\varrho,\vartheta)\,\mathrm{d}x\,\mathrm{d}t\Big).
\end{aligned}
$$

The first two terms have in accordance with (1.31)–(1.32) the good sign, while the rest can be estimated

$$
\begin{aligned}
\varepsilon m & \Big(\iint_{\{\varrho<1,\vartheta<1\}} c\,|\log\vartheta|\,\mathrm{d}x\,\mathrm{d}t + \iint_{\{\varrho>1\}} |s_0(\varrho,\vartheta)|\,\mathrm{d}x\,\mathrm{d}t\Big) \\
&\leq \varepsilon C\Big(\iint_{S^1\times\Omega} c\,|\log\vartheta|\,\mathrm{d}x\,\mathrm{d}t + \iint_{\{\varrho>1\}} \big(1+\log\varrho\big)\,\mathrm{d}x\,\mathrm{d}t\Big) \\
&\leq \varepsilon\Big(C + C\iint_{S^1\times\Omega} \frac{1}{\vartheta}\,\mathrm{d}x\,\mathrm{d}t + \iint_{S^1\times\Omega} \frac{\varrho^\gamma}{4}\,\mathrm{d}x\,\mathrm{d}t + C\,\|\vartheta\|_{L^{3B}(S^1\times\Omega)}\Big).
\end{aligned}
\tag{1.119}
$$

Thus, we get with C independent of the approximate parameters

$$
\begin{aligned}
&\int_{S^1}\int_\Omega \big(\kappa(\vartheta)+\delta\vartheta^B+\delta\vartheta^{-1}\big)\frac{|\nabla\vartheta|^2}{\vartheta^2}\,\mathrm{d}x\,\mathrm{d}t + \tau\int_{S^1}\int_\Omega \Phi'(\log\vartheta)\frac{(\partial_t\vartheta)^2}{\vartheta^3}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \ell\int_{S^1}\int_{\partial\Omega}\frac{1}{\vartheta}h(x,\vartheta)\Theta_0\,\mathrm{d}S\,\mathrm{d}t + \ell\int_{S^1}\int_\Omega\Big(\frac{1}{\vartheta}\mathbb{S}(\vartheta,\nabla\mathbf{v}_N):\nabla\mathbf{v}_N + \frac{\delta}{\vartheta^2}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \varepsilon\delta\ell\int_{S^1}\int_\Omega\frac{1}{\vartheta}(\Gamma\varrho^{\Gamma-2}+2)\,|\nabla\varrho|^2\,\mathrm{d}x\,\mathrm{d}t \leq C\Big(1+\ell\int_{S^1}\int_{\partial\Omega} h(x,\vartheta)\Theta_0\,\mathrm{d}S\,\mathrm{d}t \\
&\qquad\qquad + \tau\int_{S^1}\int_\Omega\frac{\Phi(\log\vartheta)}{\vartheta}\,\mathrm{d}x\,\mathrm{d}t + \varepsilon\ell\int_{S^1}\int_\Omega \varrho s(\varrho,\vartheta)\,\mathrm{d}x\,\mathrm{d}t\Big).
\end{aligned}
\tag{1.120}
$$

If we sum up the energy inequality (1.112), and the entropy inequality (1.120) we get

$$
\begin{aligned}
&\int_{S^1}\int_\Omega \big(\kappa(\vartheta)+\delta\vartheta^B+\delta\vartheta^{-1}\big)\frac{|\nabla\vartheta|^2}{\vartheta^2}\,\mathrm{d}x\,\mathrm{d}t + \tau\int_{S^1}\int_\Omega \Phi'(\log\vartheta)\frac{(\partial_t\vartheta)^2}{\vartheta^3}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \ell\int_{S^1}\int_{\partial\Omega} h(x,\vartheta)\left(\vartheta+\frac{\Theta_0}{\vartheta}\right)\mathrm{d}S\,\mathrm{d}t + \varepsilon\delta\ell\int_{S^1}\int_\Omega\Big(\frac{\Gamma}{\Gamma-1}\varrho^\Gamma + 2\varrho^2\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{S^1}\int_\Omega\Big(\frac{1}{\vartheta}\mathbb{S}(\vartheta,\nabla\mathbf{v}_N):\nabla\mathbf{v}_N + \frac{\ell\delta}{\vartheta^2}\Big)\,\mathrm{d}x\,\mathrm{d}t + \tau\int_{S^1}\int_\Omega \Phi(\log\vartheta)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \varepsilon\delta\ell\int_{S^1}\int_\Omega\frac{1}{\vartheta}(\Gamma\varrho^{\Gamma-2}+2)\,|\nabla\varrho|^2\,\mathrm{d}x\,\mathrm{d}t \leq C\Big(1+\ell\int_{S^1}\int_{\partial\Omega} h(x,\vartheta)\Theta_0\,\mathrm{d}S\,\mathrm{d}t \\
&\qquad\qquad + \tau\int_{S^1}\int_\Omega\frac{\Phi(\log\vartheta)}{\vartheta}\,\mathrm{d}x\,\mathrm{d}t + \ell\int_{S^1}\int_\Omega\big(\delta\vartheta^{-1}+\varrho\mathbf{g}\cdot\mathbf{v}_N\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad\qquad + \varepsilon\ell\int_{S^1}\int_\Omega \varrho s(\varrho,\vartheta)\,\mathrm{d}x\,\mathrm{d}t\Big).
\end{aligned}
\tag{1.121}
$$

The first three terms on the right-hand side can be treated using their counterparts on the left-hand side, while the last one can be estimated from (1.29) as follows

$$
\begin{aligned}
\varepsilon \int_{S^1}\int_\Omega \left(\varrho s_0(\varrho,\vartheta) + \frac{4a}{3}\vartheta^3\right) \mathrm{d}x\,\mathrm{d}t &\le \frac{\varepsilon\delta}{4}\int_{S^1}\int_\Omega \varrho^2 \,\mathrm{d}x\,\mathrm{d}t \\
&+ \frac{1}{2}\int_{S^1}\int_{\partial\Omega} h(x,\vartheta)\left(\vartheta + \frac{\Theta_0}{\vartheta}\right)\mathrm{d}S\,\mathrm{d}t + \int_{S^1}\int_\Omega \kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2}\,\mathrm{d}x\,\mathrm{d}t + C
\end{aligned}
\tag{1.122}
$$

yielding

$$
\begin{aligned}
&\int_{S^1}\int_\Omega \left(\kappa(\vartheta) + \delta\vartheta^B + \delta\vartheta^{-1}\right)\frac{|\nabla\vartheta|^2}{\vartheta^2}\,\mathrm{d}x\,\mathrm{d}t + \tau\int_{S^1}\int_\Omega \Phi'(\log\vartheta)\frac{(\partial_t\vartheta)^2}{\vartheta^3}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \ell\int_{S^1}\int_{\partial\Omega} h(x,\vartheta)\left(\vartheta + \frac{\Theta_0}{\vartheta}\right)\mathrm{d}S\,\mathrm{d}t + \varepsilon\delta\ell\int_{S^1}\int_\Omega \left(\frac{\Gamma}{\Gamma-1}\varrho^\Gamma + 2\varrho^2\right)\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{S^1}\int_\Omega \left(\frac{1}{\vartheta}\mathbb{S}(\vartheta,\nabla\mathbf{v}_N):\nabla\mathbf{v}_N + \frac{\ell\delta}{\vartheta^2}\right)\mathrm{d}x\,\mathrm{d}t + \tau\int_{S^1}\int_\Omega \Phi(\log\vartheta)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \varepsilon\delta\ell\int_{S^1}\int_\Omega \frac{1}{\vartheta}(\Gamma\varrho^{\Gamma-2}+2)\,|\nabla\varrho|^2\,\mathrm{d}x\,\mathrm{d}t \le C\left(1 + \ell\int_{S^1}\int_\Omega |\varrho\mathbf{g}\cdot\mathbf{v}_N|\,\mathrm{d}x\,\mathrm{d}t\right).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\int_{S^1}\int_\Omega |\varrho\mathbf{g}\cdot\mathbf{v}_N|\,\mathrm{d}x\,\mathrm{d}t &\le C\,\|\mathbf{v}_N\|_{L^2(L^6)}\,\|\varrho\|_{L^2\left(L^{6/5}\right)} \\
&\le \frac{1}{2}\int_{S^1}\int_\Omega \left(\frac{1}{\vartheta}\mathbb{S}(\vartheta,\nabla\mathbf{v}_N):\nabla\mathbf{v}_N\right)\mathrm{d}x\,\mathrm{d}t + C\,\|\varrho\|^2_{L^2\left(L^{6/5}\right)} \\
&\le \frac{1}{2}\int_{S^1}\int_\Omega \left(\frac{1}{\vartheta}\mathbb{S}(\vartheta,\nabla\mathbf{v}_N):\nabla\mathbf{v}_N\right)\mathrm{d}x\,\mathrm{d}t + \frac{\varepsilon\delta}{4}\int_{S^1}\int_\Omega \varrho^2\,\mathrm{d}x\,\mathrm{d}t + C(\varepsilon,\delta).
\end{aligned}
\tag{1.123}
$$

Consequently, we have proved the following counterpart of Lemma 5 from [44].

Proposition 1.10. *Any solution to (1.104)* $(\mathbf{v}_N,\vartheta)$ *satisfies:*

$$
\begin{aligned}
&\int_{S^1}\int_\Omega \left(\kappa(\vartheta) + \delta\vartheta^B + \delta\vartheta^{-1}\right)\frac{|\nabla\vartheta|^2}{\vartheta^2}\,\mathrm{d}x\,\mathrm{d}t + \tau\int_{S^1}\int_\Omega \Phi'(\log\vartheta)\frac{(\partial_t\vartheta)^2}{\vartheta^3}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \ell\int_{S^1}\int_{\partial\Omega} h(x,\vartheta)\left(\vartheta + \frac{\Theta_0}{\vartheta}\right)\mathrm{d}S\,\mathrm{d}t + \varepsilon\delta\ell\int_{S^1}\int_\Omega \left(\frac{\Gamma}{\Gamma-1}\varrho^\Gamma + 2\varrho^2\right)\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{S^1}\int_\Omega \left(\frac{1}{\vartheta}\mathbb{S}(\vartheta,\nabla\mathbf{v}_N):\nabla\mathbf{v}_N + \frac{\ell\delta}{\vartheta^2}\right)\mathrm{d}x\,\mathrm{d}t + \tau\int_{S^1}\int_\Omega \Phi(\log\vartheta)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \varepsilon\delta\ell\int_{S^1}\int_\Omega \frac{1}{\vartheta}(\Gamma\varrho^{\Gamma-2}+2)\,|\nabla\varrho|^2\,\mathrm{d}x\,\mathrm{d}t \le C(\varepsilon,\delta),
\end{aligned}
\tag{1.124}
$$

where the constant on the right-hand side is independent of $\tau, N,$ *as well as* ζ, *and* ϱ *is given by Proposition 1.7 as* $\varrho = \varrho(\mathbf{v}_N)$.

By virtue of Proposition 1.10, we can apply Theorem 4.9 on mapping $\mathcal{T}$, which completes the proof of Lemma 1.6. Recalling that the solution to our problem fulfils (1.104) with $\ell = 1$, inequality (1.124) holds for the solution from Lemma 1.6 with $\ell = 1$.

1.4 Limit passages

1.4.1 Limits $\tau \to 0^+$ and $N \to \infty$

Now, we will perform the limit passage as $\tau \to 0^+$. From (1.124) we have

$$\|\nabla \mathbf{v}_N\|_{L^2(L^2)} + \|\nabla \vartheta\|_{L^2(L^2)} + \left\|\nabla \vartheta^{B/2}\right\|_{L^2(L^2)} + \|\vartheta\|_{L^1(L^{3B})} + \left\|\vartheta^4 + \vartheta^{-1}\right\|_{L^1(S^1\times\partial\Omega)} \\ + \left\|\nabla \vartheta^{-1/2}\right\|_{L^2(L^2)} + \left\|\vartheta^{-1}\right\|_{L^2(L^2)} + \|\varrho\|_{L^\Gamma(L^{3\Gamma})} \le C(\varepsilon,\delta) \quad (1.125)$$

and from (1.93)

$$\left\|\nabla^2 \varrho\right\|_{L^q(L^q)} + \|\partial_t \varrho\|_{L^q(L^q)} \le C(\varepsilon,\delta), \quad (1.126)$$

with some $q \in (1,2)$. Moreover, since the velocity belongs to a finite-dimensional space, it is obviously relatively compact.

We can also test the energy equation by $\Phi(\log\vartheta)$ and $\partial_t\Phi(\log\vartheta)$, in order to get some additional information depending on ζ, namely

$$\tau \left\|\frac{\partial \Phi}{\partial t}(\log\vartheta)\right\|^2_{L^2(L^2)} + \tau \|\Phi(\log\vartheta)\|^2_{L^2(L^2)} + \|\nabla\Phi(\log\vartheta)\|^2_{L^2(L^2)} \\ + \zeta \left\|\Phi'(\log\vartheta)\left(\frac{(\partial_t\vartheta)^2}{\vartheta} + \vartheta(\partial_t\vartheta)^2\right)\right\|_{L^1(L^1)} \le C(\varepsilon,\delta,N). \quad (1.127)$$

Further, we have $\Phi'(\log\vartheta)\vartheta^{-1} \ge K > 0$, which yields using the structure of Φ

$$\zeta \|\partial_t\vartheta\|_{L^2(L^2)} + \|\nabla\vartheta\|_{L^2(L^2)} \le C(\varepsilon,\delta,N). \quad (1.128)$$

To summarize, we have according to Theorem 4.6 the strong and pointwise convergence of the temperature inside the domain, and we can easily pass to the limit as $\tau \to 0^+$ there. However, the issue of convergence of the temperature in the nonlinear terms on the boundary requires some more attention. Note that the most restrictive nonlinear terms are $h(x,\vartheta)\vartheta$ in (1.96) and $h(x,\vartheta)\Theta_0/\vartheta$ in (1.97) for which the basic estimate (1.125) ensures directly L^1-integrability only. Nevertheless, besides the interpolation (1.92) we have analogously

$$\left\|\vartheta^{-1}\right\|^{3/2}_{L^{3/2}(S^1\times\partial\Omega)} \le C\int_{S^1} \left\|\nabla(\vartheta^{-1/2})\right\|_{L^2(\Omega)} \left\|\vartheta^{-1}\right\|_{L^2(\Omega)} \mathrm{d}t \\ \le C\left\|\vartheta^{-1/2}\right\|_{L^2(W^{1,2}(\Omega))} \left\|\vartheta^{-1}\right\|_{L^2(S^1\times\Omega)} \le C, \quad (1.129)$$

so according to the compactness of the trace operator we can identify all of the nonlinear terms on the boundary. Thus, we get at least one solution in the class $\varrho \in W^{1,p}(S^1\times\Omega) \cap L^p(S^1, W^{2,p}(\Omega))$, $\mathbf{v}_N \in Lin\{\mathbf{w}^i\}_{i=1}^N$, $\log\vartheta \in W^{1,p}(S^1\times\Omega) \cap L^p(S^1, W^{2,p}(\Omega))$ satisfying (1.93), (1.94), (1.96), (1.97) with $\tau = 0$. Moreover, we have (1.124) with $\tau = 0$ and $\ell = 1$, and the following bound

$$\int_{S^1}\int_\Omega \left(\left(\kappa(\vartheta) + \delta\vartheta^B + \delta\vartheta^{-1}\right)\frac{|\nabla\vartheta|^2}{\vartheta^2} + \frac{1}{\vartheta}\mathbb{S}(\vartheta,\nabla\mathbf{v}_N) : \nabla\mathbf{v}_N + \frac{\delta}{\vartheta^2}\right)\mathrm{d}x\,\mathrm{d}t \\ + \varepsilon\delta\int_{S^1}\int_\Omega \frac{1}{\vartheta}(\Gamma\varrho^{\Gamma-2}+2)\,|\nabla\varrho|^2\,\mathrm{d}x\,\mathrm{d}t + \int_{S^1}\int_{\partial\Omega} h(x,\vartheta)\frac{\Theta_0}{\vartheta}\,\mathrm{d}S\,\mathrm{d}t \\ \le C\left(1 + \varepsilon\int_{S^1}\int_\Omega \varrho s(\varrho,\vartheta)\,\mathrm{d}x\,\mathrm{d}t\right), \quad (1.130)$$

with C independent of N, ζ, δ, and ε.

Concerning the limit in the Galerkin approximation, we will pass first to the limit in the momentum equation. However, our approach requires to split this passage into two steps. Let us recall that we consider the basis function of the finite dimensional space in a special form $\mathbf{w}^i(t,x) = a^k(t)\mathbf{b}^l(x)$, where $1 \le i \le N$ corresponds to $1 \le k \le N_t$, and $1 \le l \le N_x$, so we denote $\mathbf{v}_N = \mathbf{v}_{N_t,\,N_x}$. First, let $N_t \to \infty$, the bound (1.124) yields that we can find a subsequence (denoted in the same way) such that

$$\mathbf{v}_{N_t,\,N_x} \rightharpoonup \mathbf{v}_{N_x} \text{ weakly in } L^2\left(S^1, \mathbb{R}^{3N_x}\right) \text{ as } N_t \to \infty.$$

From (1.94) we can get the "dual" estimate of $\frac{\partial \mathbf{v}_N}{\partial t}$ in $L^q\left(S^1, \mathbb{R}^{3N_x}\right)$, for some $q > 1$, and consequently also of $\mathbf{v}_N$ in $L^\infty\left(S^1, \mathbb{R}^{3N_x}\right)$. Going again back to the momentum equation we obtain $\frac{\partial \mathbf{v}_N}{\partial t}$ in $L^q\left(S^1, \mathbb{R}^{3N_x}\right)$, for any $q < \infty$. Since the time derivatives are the only problematic terms in this passage we immediately get

$$\begin{aligned}
&\int_{S^1} \phi \int_\Omega \left(\zeta \frac{\partial \mathbf{v}_{N_x}}{\partial t} \cdot \mathbf{b}^l + \frac{\partial(\varrho \mathbf{v}_{N_x})}{\partial t} \cdot \mathbf{b}^l - (\varrho \mathbf{v}_{N_x} \otimes \mathbf{v}_{N_x}) : \nabla \mathbf{b}^l \right) \mathrm{d}x \, \mathrm{d}t \\
&\qquad + \int_{S^1} \phi \int_\Omega \left(\mathbb{S}(\vartheta, \nabla \mathbf{v}_{N_x}) : \nabla \mathbf{b}^l - \left(p(\varrho,\vartheta) + \delta(\varrho^\Gamma + \varrho^2)\right) \operatorname{div} \mathbf{b}^l \right) \mathrm{d}x \, \mathrm{d}t \\
&= \int_{S^1} \phi \int_\Omega \left(-\varepsilon(\nabla \varrho \cdot \nabla \mathbf{v}_{N_x}) \cdot \mathbf{b}^l + \frac{1}{2}\varepsilon(m - \varrho)\mathbf{v}_{N_x} \cdot \mathbf{b}^l + \varrho \mathbf{g} \cdot \mathbf{b}^l \right) \mathrm{d}x \, \mathrm{d}t \quad (1.131)
\end{aligned}$$

for all $\phi \in C^\infty(S^1)$, and $l = 1, \dots, N_x$.

The next step is to pass to the limit in the space approximation; we will write $\mathbf{v}_{N_x} = \mathbf{v}_N$ and investigate the limit as $N_x \to \infty$.

First we want to obtain a total energy estimate, for this purpose we use $\mathbf{v}_N\psi$, for any $\psi \in C^\infty(S^1)$, as a test function in (1.94) yielding

$$\begin{aligned}
&-\int_{S^1} \left(\int_\Omega \left(\zeta \frac{|\mathbf{v}_N|^2}{2} + \varrho \frac{|\mathbf{v}_N|^2}{2} \right) \mathrm{d}x \frac{\partial \psi}{\partial t} \right) \mathrm{d}t + \int_{S^1} \int_\Omega \left(\mathbb{S}(\vartheta, \nabla \mathbf{v}_N) : \nabla \mathbf{v}_N \right) \mathrm{d}x \, \psi \, \mathrm{d}t \\
&\qquad = \int_{S^1} \int_\Omega \left(\left(p(\varrho,\vartheta) + \delta(\varrho^\Gamma + \varrho^2)\right) \operatorname{div} \mathbf{v}_N + \varrho \mathbf{g} \cdot \mathbf{v}_N \right) \mathrm{d}x \, \psi \, \mathrm{d}t. \quad (1.132)
\end{aligned}$$

Next we integrate the energy equation (1.96) (with $\tau = 0$) over Ω, multiply by a function $\psi \in C^\infty(S^1)$, and integrate over S^1 to get

$$\begin{aligned}
&-\int_{S^1} \left(\int_\Omega \left(\zeta\vartheta + \varrho e(\varrho,\vartheta) \right) \mathrm{d}x \right) \frac{\partial \psi}{\partial t} \, \mathrm{d}t + \int_{S^1} \int_{\partial\Omega} h(x,\vartheta)(\vartheta - \Theta_0) \, \mathrm{d}S \, \psi \, \mathrm{d}t \\
&\qquad = \int_{S^1} \int_\Omega \left(\mathbb{S}(\vartheta, \nabla \mathbf{v}_N) : \nabla \mathbf{v}_N - p(\varrho,\vartheta) \operatorname{div} \mathbf{v}_N \right) \mathrm{d}x \, \psi \, \mathrm{d}t \\
&\qquad\qquad + \int_{S^1} \int_\Omega \left(\frac{\delta}{\vartheta} + \delta\varepsilon \, |\nabla \varrho|^2 \left(\Gamma \varrho^{\Gamma-2} + 2\right) \right) \mathrm{d}x \, \psi \, \mathrm{d}t. \quad (1.133)
\end{aligned}$$

The last ingredient for the total energy balance is the renormalized version of the continuity equation (1.93), which reads

$$\begin{aligned}
&-\int_{S^1} \int_\Omega \left(\frac{\varrho^\beta}{\beta - 1} \right) \mathrm{d}x \frac{\partial \psi}{\partial t} \, \mathrm{d}t + \varepsilon\beta \int_{S^1} \int_\Omega \left(\frac{\varrho^\beta}{\beta - 1} + \varrho^{\beta-2} \, |\nabla \varrho|^2 \right) \mathrm{d}x \, \psi \, \mathrm{d}t \\
&\qquad + \int_{S^1} \int_\Omega \varrho^\beta \operatorname{div} \mathbf{v} \, \mathrm{d}x \, \psi \, \mathrm{d}t = \frac{\varepsilon\beta}{\beta - 1} \int_{S^1} \int_\Omega m \varrho^{\beta-1} \, \mathrm{d}x \, \psi \, \mathrm{d}t. \quad (1.134)
\end{aligned}$$

Thus, if we sum up (1.132), (1.133), and (1.134) for $\beta = 2$, Γ multiplied by δ, we get

$$
\begin{aligned}
-\int_{S^1}\int_\Omega \Big((\zeta+\varrho)\frac{|\mathbf{v}_N|^2}{2} + \zeta\vartheta + \varrho e(\varrho,\vartheta) + \delta\big(\frac{\varrho^\Gamma}{\Gamma-1} + \varrho^2\big)\Big)\, \mathrm{d}x \frac{\partial \psi}{\partial t}\, \mathrm{d}t \\
+ \int_{S^1}\int_{\partial\Omega} h(x,\vartheta)(\vartheta - \Theta_0)\, \mathrm{d}S\, \psi\ \mathrm{d}t + \varepsilon\delta \int_{S^1}\int_\Omega \Big(\frac{\Gamma\varrho^\Gamma}{\Gamma-1} + 2\varrho^2\Big)\, \mathrm{d}x\, \psi\ \mathrm{d}t \\
= \int_{S^1}\int_\Omega \Big(\varrho\mathbf{g}\cdot\mathbf{v}_N + \varepsilon\delta\frac{\Gamma}{\Gamma-1} m\varrho^{\Gamma-1} + 2\varepsilon\delta m\varrho + \frac{\delta}{\vartheta}\Big)\, \mathrm{d}x\, \psi\ \mathrm{d}t. \quad (1.135)
\end{aligned}
$$

Further, integrating the entropy equation (1.97) (again with $\tau = 0$) over Ω, multiplying by a function $\psi \in C^\infty(S^1)$, and integrating over S^1 after some integration by parts with use of the boundary conditions we end up with

$$
\begin{aligned}
\int_{S^1}\int_\Omega \big(\zeta\log\vartheta + \varrho s(\varrho,\vartheta)\big)\, \mathrm{d}x \frac{\partial\psi}{\partial t}\, \mathrm{d}t + \int_{S^1}\int_{\partial\Omega} \frac{1}{\vartheta} h(x,\vartheta)\Theta_0\, \mathrm{d}S\, \psi\ \mathrm{d}t \\
+ \int_{S^1}\int_\Omega \Big(\big(\kappa(\vartheta) + \delta\vartheta^B + \delta\vartheta^{-1}\big)\frac{|\nabla\vartheta|^2}{\vartheta^2}\Big)\, \mathrm{d}x\, \psi\ \mathrm{d}t \\
+ \int_{S^1}\int_\Omega \Big(\frac{1}{\vartheta}\mathbb{S}(\vartheta,\nabla\mathbf{v}_N) : \nabla\mathbf{v}_N + \frac{\delta}{\vartheta^2}\Big)\, \mathrm{d}x\, \psi\ \mathrm{d}t \\
+ \varepsilon\delta \int_{S^1}\int_\Omega \Big(\frac{1}{\vartheta}(\Gamma\varrho^{\Gamma-2} + 2)\, |\nabla\varrho|^2\Big)\, \mathrm{d}x\, \psi\ \mathrm{d}t = \int_{S^1}\int_{\partial\Omega} h(x,\vartheta)\Theta_0\ \mathrm{d}S\, \psi\ \mathrm{d}t \\
+ \int_{S^1}\int_\Omega \Big(\big(\operatorname{div}(\varrho\mathbf{v}_N) + \partial_t\varrho\big)\frac{\varrho e + p - \varrho\vartheta s}{\varrho\vartheta}\Big)\, \mathrm{d}x\, \psi\ \mathrm{d}t.
\end{aligned}
$$

Estimating the last integral on the right-hand side in the same manner as between (1.114) and (1.119) yields

$$
\begin{aligned}
\int_{S^1}\int_\Omega \big(\zeta\log\vartheta + \varrho s(\varrho,\vartheta)\big)\, \mathrm{d}x \frac{\partial\psi}{\partial t}\, \mathrm{d}t + \int_{S^1}\int_{\partial\Omega} \frac{1}{\vartheta} h(x,\vartheta)\Theta_0\, \mathrm{d}S\, \psi\ \mathrm{d}t \\
+ \int_{S^1}\int_\Omega \Big(\big(\kappa(\vartheta) + \delta\vartheta^B + \delta\vartheta^{-1}\big)\frac{|\nabla\vartheta|^2}{\vartheta^2}\Big)\, \mathrm{d}x\, \psi\ \mathrm{d}t \\
+ \int_{S^1}\int_\Omega \Big(\frac{1}{\vartheta}\mathbb{S}(\vartheta,\nabla\mathbf{v}_N) : \nabla\mathbf{v}_N + \frac{\delta}{\vartheta^2}\Big)\, \mathrm{d}x\, \psi\ \mathrm{d}t \\
+ \frac{\varepsilon\delta}{2} \int_{S^1}\int_\Omega \Big(\frac{1}{\vartheta}(\Gamma\varrho^{\Gamma-2} + 2)\, |\nabla\varrho|^2\Big)\, \mathrm{d}x\, \psi\ \mathrm{d}t \\
\le C\Big(1 + \varepsilon\int_{S^1}\int_\Omega \varrho s(\varrho,\vartheta)\, \mathrm{d}x\, \psi\ \mathrm{d}t + \int_{S^1}\int_{\partial\Omega} h(x,\vartheta)\Theta_0\ \mathrm{d}S\, \psi\ \mathrm{d}t\Big). \\
(1.136)
\end{aligned}
$$

If we add the total energy balance (1.135) to the estimate coming from entropy equality (1.136), and put the terms containing $h(x,\vartheta)\Theta_0$, $\varrho^{\beta-1}$, and ϑ^{-1} to the left-hand side using Young's inequality, we obtain after some reordering

$$
\begin{aligned}
&-\int_{S^1}\int_\Omega \left((\zeta+\varrho)\frac{|\mathbf{v}_N|^2}{2} + \zeta(\vartheta - \log\vartheta) + \varrho(e-s) + \delta\Big(\frac{\varrho^\Gamma}{\Gamma-1} + \varrho^2\Big)\right) \mathrm{d}x \frac{\partial\psi}{\partial t}\,\mathrm{d}t \\
&\quad + \int_{S^1}\int_\Omega \left(\big(\kappa(\vartheta) + \delta\vartheta^B + \delta\vartheta^{-1}\big)\frac{|\nabla\vartheta|^2}{\vartheta^2} + \frac{\delta}{2\vartheta^2}\right) \mathrm{d}x\,\psi\,\mathrm{d}t \\
&\quad + \int_{S^1}\int_\Omega \Big(\frac{1}{\vartheta}\mathbb{S}(\vartheta,\nabla\mathbf{v}_N) : \nabla\mathbf{v}_N\Big)\,\mathrm{d}x\,\psi\,\mathrm{d}t \\
&\quad + \frac{\varepsilon\delta}{2}\int_{S^1}\int_\Omega \Big(\frac{\Gamma\varrho^\Gamma}{\Gamma-1} + 2\varrho^2 + \frac{|\nabla\varrho|^2}{\vartheta}(\Gamma\varrho^{\Gamma-2}+2)\Big)\,\mathrm{d}x\,\psi\,\mathrm{d}t \\
&\quad + \frac{1}{2}\int_{S^1}\int_{\partial\Omega} h(x,\vartheta)\left(\frac{\Theta_0}{\vartheta} + \vartheta\right)\mathrm{d}S\,\psi\,\mathrm{d}t \\
&\qquad \le C\left(1 + \left|\int_{S^1}\int_\Omega \varrho\mathbf{g}\cdot\mathbf{v}_N\,\mathrm{d}x\,\psi\,\mathrm{d}t\right| + \varepsilon\left|\int_{S^1}\int_\Omega \varrho s(\varrho,\vartheta)\,\mathrm{d}x\,\psi\,\mathrm{d}t\right|\right).
\end{aligned} \tag{1.137}
$$

Inequality (1.137) provides estimates, which are useful, but not sufficient for the compactness of the velocity and temperature in time. In order to mimic the considerations below (1.76) we define the modified total energy

$$
\mathcal{E}_N(t) = \int_\Omega \left((\zeta+\varrho)\frac{|\mathbf{v}_N|^2}{2} + \zeta(\vartheta-\log\vartheta) + H(\varrho,\vartheta) + \delta\Big(\frac{\varrho^\Gamma}{\Gamma-1}+\varrho^2\Big)\right)\mathrm{d}x, \tag{1.138}
$$

where $H(\varrho,\vartheta)$ is so-called Helmholtz function

$$
H(\varrho,\vartheta) = \varrho\big(e(\varrho,\vartheta) - s(\varrho,\vartheta)\big).
$$

Let us note that due to our structural assumptions (1.25), (1.30), (1.28) we have

$$
H(\varrho,\vartheta) = \varrho\big(e_0(\varrho,\vartheta) - s_0(\varrho,\vartheta)\big) + a\Big(\vartheta^4 - \frac{4}{3}\vartheta^3\Big) \ge c\varrho^\gamma - C\varrho\,|\log\varrho| + \frac{\vartheta^4}{2} - C \tag{1.139}
$$

and consequently also

$$
\mathcal{E}_N(t) \ge \int_\Omega \left((\zeta+\varrho)\frac{|\mathbf{v}_N|^2}{2} + \frac{\zeta}{2}(\vartheta + |\log\vartheta|) + \delta\Big(\frac{\varrho^\Gamma}{\Gamma-1}+\varrho^2\Big) + c\varrho^\gamma + \frac{\vartheta^4}{2} - C\right)\mathrm{d}x. \tag{1.140}
$$

It can be deduced from (1.137) (similarly as for (1.76)) that

$$
\sup_{t\in S^1}\mathcal{E}_N(t) \le C(\varepsilon,\delta)\left(1 + \int_{S^1}\mathcal{E}_N(t)\,\mathrm{d}t\right).
$$

All terms on the right-hand side of this inequality can be pushed to the left-hand side; let us consider here only the most difficult one,[17] id est $\varrho\,|\mathbf{v}_N|^2$.

$$
\begin{aligned}
\int_{S^1}\int_\Omega \varrho\,|\mathbf{v}_N|^2\,\mathrm{d}x\,\mathrm{d}t &\le \int_{S^1}\int_\Omega \sqrt{\varrho}\,\,|\sqrt{\varrho}\mathbf{v}_N|\,|\mathbf{v}_N|\,\mathrm{d}x\,\mathrm{d}t \\
&\le C\,\|\sqrt{\varrho}\|_{L^{2\Gamma}(S^1\times\Omega)}\,\|\sqrt{\varrho}\mathbf{v}_N\|_{L^\infty(L^2)}\,\|\mathbf{v}_N\|_{L^2(L^6)} \\
&\le C(\varepsilon,\delta)\sup_{t\in S^1}\mathcal{E}_N^{1/2}(t),
\end{aligned} \tag{1.141}
$$

[17]Note that there is no problem with the term ϑ^4, since $\vartheta \in L^3\left(S^1; L^9(\Omega)\right)$

provided $\Gamma > \frac{3}{2}$, thus we obtain

$$\sup_{t\in S^1} \mathcal{E}_N(t) \le C(\varepsilon,\delta). \tag{1.142}$$

Further, from (1.125) we have due to the Poincaré inequality

$$\|\vartheta\|_{L^B(L^{3B})} \le C(\varepsilon,\delta).$$

In order to use the Aubin-Lions lemma 4.6 we need to have some bound on the time derivative of temperature and velocity. Noting that

$$\|\mathbf{v}_N\|_{L^{10/3}(S^1\times\Omega)} \le C\,\|\mathbf{v}_N\|^{2/5}_{L^\infty(L^2)}\,\|\mathbf{v}_N\|^{3/5}_{L^2(L^6)} \le C(\varepsilon,\delta)$$

we improve by means of Theorem 4.11 the estimates coming from the continuity equation (1.93) up to

$$\|\partial_t\varrho\|_{L^{3/2}(S^1\times\Omega)} + \left\|\nabla^2\varrho\right\|_{L^{3/2}(S^1\times\Omega)} + \|\nabla\varrho\|_{L^3(S^1\times\Omega)} \le C(\varepsilon,\delta), \tag{1.143}$$

provided $\Gamma \ge 30$. Consequently, we can get from (1.94) for example the following desired "dual" estimate on the time derivative of the velocity field

$$\zeta\,\|\partial_t\mathbf{v}_N\|_{(L^{30}(S^1,\,W_0^{1,5}(\Omega)))^*} \le C(\varepsilon,\delta),$$

leading to strong convergence of velocity.

Similarly, for the temperature we get from (1.97) with $\tau = 0$

$$\left\|\Big(\frac{\zeta+\varrho}{\vartheta} + \vartheta^2\Big)\frac{\partial\vartheta}{\partial t}\right\|_{L^1(S^1,\,W_0^{1,5}(\Omega)^*)} \le C(\varepsilon,\delta).$$

Thus, we get

$$\vartheta_N \to \vartheta \text{ a.e. in } S^1\times\Omega, \tag{1.144}$$

and thanks to estimates of ϑ coming from interpolations (1.92) and (1.129) also

$$\vartheta_N \to \vartheta \text{ in } L^4(S^1\times\partial\Omega), \qquad \vartheta_N^{-1} \to \vartheta^{-1} \text{ in } L^1(S^1\times\partial\Omega). \tag{1.145}$$

Considering the modified pressure we can interpolate

$$\|\varrho\|_{L^{5\Gamma/3}(S^1\times\Omega)} \le \|\varrho\|^{2/5}_{L^\infty(L^\Gamma)}\,\|\varrho\|^{3/5}_{L^\Gamma(L^{3\Gamma})} \le C(\varepsilon,\delta), \tag{1.146}$$

where the right-hand side is bounded according to (1.142) and (1.125), so we converge strongly in the term ϱ^Γ. To conclude, we can pass to the limit with $N\to\infty$ in continuity equation (1.93) as well as in the momentum equation (1.94).

The most delicate term in the limit $N\to\infty$ is $\mathbb{S}(\vartheta,\nabla\mathbf{v}_N):\nabla\mathbf{v}_N$ in the energy equation (1.96), for which we are not even able to guarantee the boundedness in L^1 space. Therefore, from now we consider only the entropy equation, since the corresponding term $\frac{1}{\vartheta}\mathbb{S}(\vartheta,\nabla\mathbf{v}_N):\nabla\mathbf{v}_N$, which appears in (1.97), is bounded in $L^1(S^1\times\Omega)$. Thus, in the entropy equation, considering the terms of the entropy production we have only the following weak pieces of information

$$\begin{aligned}&\frac{\mathbb{S}(\vartheta,\nabla\mathbf{v}_N):\nabla\mathbf{v}_N}{\vartheta_N},\ \big(\kappa(\vartheta_N)+\delta\vartheta_N^B+\delta\vartheta_N^{-1}\big)\frac{|\nabla\vartheta_N|^2}{\vartheta_N^2},\\ &\qquad\frac{\mathbb{S}(\vartheta,\nabla\mathbf{v}):\nabla\mathbf{v}}{\vartheta},\ \big(\kappa(\vartheta)+\delta\vartheta^B+\delta\vartheta^{-1}\big)\frac{|\nabla\vartheta|^2}{\vartheta^2} \in L^1(S^1\times\Omega),\end{aligned} \tag{1.147}$$

which allows us to get only the weak convergence in $\mathcal{M}(S^1 \times \Omega)$

$$\frac{\mathbb{S}(\vartheta, \nabla \mathbf{v}_N) : \nabla \mathbf{v}_N}{\vartheta_N} + \big(\kappa(\vartheta_N) + \delta \vartheta_N^B + \delta \vartheta_N^{-1}\big) \frac{|\nabla \vartheta_N|^2}{\vartheta_N^2} \rightharpoonup \sigma, \tag{1.148}$$

where in addition, since norm is weakly lower semicontinuous σ satisfies in accordance with Theorem 4.18

$$\sigma \geq \frac{\mathbb{S}(\vartheta, \nabla \mathbf{v}) : \nabla \mathbf{v}}{\vartheta} + \big(\kappa(\vartheta) + \delta \vartheta^B + \delta \vartheta^{-1}\big) \frac{|\nabla \vartheta|^2}{\vartheta^2} \geq 0. \tag{1.149}$$

Note that as long as $\delta > 0$ the boundary term with temperature in the entropy equation is compact, so at this stage we do not have to treated it as σ. Finally, with (1.145) in hands the limit in the total energy balance (1.135) is straightforward.

Let us now summarize what we have so far proved.

Proposition 1.11. *There exists at least one approximative solution* $(\varrho, \mathbf{v}, \vartheta)$ *such that* $\varrho \in W^{1,3/2}(S^1 \times \Omega) \cap L^{3/2}(S^1; W^{2,p}(\Omega))$, $\mathbf{v} \in L^2(S^1; W_0^{1,2}(\Omega))$ *and* $\vartheta \in L^2(S^1; W^{1,2}(\Omega))$ *satisfying*

$$\begin{aligned} \frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{v}) - \varepsilon \Delta \varrho + \varepsilon \varrho = \varepsilon m \ \textit{in}\ S^1 \times \Omega, \\ \frac{\partial \varrho}{\partial \mathbf{n}} = 0 \ \textit{on}\ S^1 \times \partial\Omega, \end{aligned} \tag{1.150}$$

for all $\boldsymbol{\varphi} \in C_0^\infty(S^1 \times \Omega; \mathbb{R}^3)$

$$\begin{aligned} &\int_{S^1} \int_\Omega \Big(\zeta \frac{\partial \mathbf{v}}{\partial t} \cdot \boldsymbol{\varphi} + \frac{\partial (\varrho \mathbf{v})}{\partial t} \cdot \boldsymbol{\varphi} - (\varrho \mathbf{v} \otimes \mathbf{v}) : \nabla \boldsymbol{\varphi} \Big) \,\mathrm{d}x \,\mathrm{d}t \\ &\quad + \int_{S^1} \int_\Omega \Big(\mathbb{S}(\vartheta, \nabla \mathbf{v}) : \nabla \boldsymbol{\varphi} - \big(p(\varrho, \vartheta) + \delta(\varrho^\Gamma + \varrho^2)\big) \operatorname{div} \boldsymbol{\varphi} \Big) \,\mathrm{d}x \,\mathrm{d}t \\ &\quad = \int_{S^1} \int_\Omega \Big(-\varepsilon \nabla \varrho \cdot \nabla \mathbf{v} \cdot \boldsymbol{\varphi} + \frac{1}{2} \varepsilon (m - \varrho) \mathbf{v} \cdot \boldsymbol{\varphi} + \varrho \mathbf{g} \cdot \boldsymbol{\varphi} \Big) \,\mathrm{d}x \,\mathrm{d}t, \end{aligned} \tag{1.151}$$

for all $\psi \in C^\infty(S^1 \times \overline{\Omega})$

$$\begin{aligned} &\int_{S^1} \int_\Omega \Big(-\big(\zeta \log \vartheta + \varrho s(\varrho, \vartheta)\big) \frac{\partial \psi}{\partial t} - \varrho s(\varrho, \vartheta) \mathbf{v} \cdot \nabla \psi \Big) \,\mathrm{d}x \,\mathrm{d}t \\ &\quad + \int_{S^1} \int_\Omega \Big(\operatorname{div}(\varrho \mathbf{v}) + \frac{\partial \varrho}{\partial t} \Big) \frac{\varrho e + p - \varrho \vartheta s}{\varrho \vartheta} \psi \,\mathrm{d}x \,\mathrm{d}t \\ &\quad + \int_{S^1} \int_\Omega \Big(\big(\kappa(\vartheta) + \delta \vartheta^B + \delta \vartheta^{-1}\big) \frac{\nabla \vartheta}{\vartheta} \cdot \nabla \psi \Big) \,\mathrm{d}x \,\mathrm{d}t \\ &\quad + \int_{S^1} \int_{\partial\Omega} \frac{h(\vartheta - \Theta_0)}{\vartheta} \psi \,\mathrm{d}S \,\mathrm{d}t = \langle \sigma, \psi \rangle + \delta \int_{S^1} \int_\Omega \frac{1}{\vartheta^2} \psi \,\mathrm{d}x \,\mathrm{d}t \\ &\qquad\qquad + \delta \varepsilon \int_{S^1} \int_\Omega \frac{1}{\vartheta} (\Gamma \varrho^{\Gamma - 2} + 2) \, |\nabla \varrho|^2 \, \psi \,\mathrm{d}x \,\mathrm{d}t, \end{aligned} \tag{1.152}$$

with

$$\sigma \geq \frac{\mathbb{S}(\vartheta, \nabla \mathbf{v}) : \nabla \mathbf{v}}{\vartheta} + \big(\kappa(\vartheta) + \delta \vartheta^B + \delta \vartheta^{-1}\big) \frac{|\nabla \vartheta|^2}{\vartheta^2} \geq 0, \tag{1.153}$$

and for all $\psi \in C^\infty(S^1)$

$$
\begin{aligned}
-\int_{S^1}\int_\Omega \Big((\zeta+\varrho)\frac{|\mathbf{v}|^2}{2} + \zeta\vartheta + \varrho e(\varrho,\vartheta) + \delta\big(\frac{\varrho^\Gamma}{\Gamma-1}+\varrho^2\big)\Big)\,\mathrm{d}x\frac{\partial\psi}{\partial t}\,\mathrm{d}t \\
+\int_{S^1}\int_{\partial\Omega} h(x,\vartheta)(\vartheta-\Theta_0)\,\mathrm{d}S\,\psi\,\,\mathrm{d}t + \varepsilon\delta\int_{S^1}\int_\Omega\Big(\frac{\Gamma\varrho^\Gamma}{\Gamma-1}+2\varrho^2\Big)\,\mathrm{d}x\,\psi\,\,\mathrm{d}t \\
= \int_{S^1}\int_\Omega\Big(\varrho\mathbf{g}\cdot\mathbf{v} + \varepsilon\delta\frac{\Gamma}{\Gamma-1}m\varrho^{\Gamma-1} + 2\varepsilon\delta m\varrho + \frac{\delta}{\vartheta}\Big)\,\mathrm{d}x\,\psi\,\,\mathrm{d}t. \quad (1.154)
\end{aligned}
$$

Moreover, we have for $\psi \in C^\infty(S^1)$ non-negative

$$
\begin{aligned}
\int_{S^1}\int_\Omega\big(\zeta\log\vartheta + \varrho s(\varrho,\vartheta)\big)\,\mathrm{d}x\frac{\partial\psi}{\partial t}\,\mathrm{d}t + \int_{S^1}\int_{\partial\Omega}\frac{1}{\vartheta}h(x,\vartheta)\Theta_0\,\mathrm{d}S\,\psi\,\,\mathrm{d}t \\
+\int_{S^1}\int_\Omega\Big(\big(\kappa(\vartheta)+\delta\vartheta^B+\delta\vartheta^{-1}\big)\frac{|\nabla\vartheta|^2}{\vartheta^2}\Big)\,\mathrm{d}x\,\psi\,\,\mathrm{d}t \\
+\int_{S^1}\int_\Omega\Big(\frac{1}{\vartheta}\mathbb{S}(\vartheta,\nabla\mathbf{v}):\nabla\mathbf{v} + \frac{\delta}{\vartheta^2}\Big)\,\mathrm{d}x\,\psi\,\,\mathrm{d}t \\
+\frac{\varepsilon\delta}{2}\int_{S^1}\int_\Omega\Big(\frac{1}{\vartheta}(\Gamma\varrho^{\Gamma-2}+2)\,|\nabla\varrho|^2\Big)\,\mathrm{d}x\,\psi\,\,\mathrm{d}t \\
\le C\Big(1+\varepsilon\int_{S^1}\int_\Omega\varrho s(\varrho,\vartheta)\,\mathrm{d}x\,\psi\,\,\mathrm{d}t + \int_{S^1}\int_{\partial\Omega}h(x,\vartheta)\Theta_0\,\,\mathrm{d}S\,\psi\,\,\mathrm{d}t\Big), \\
(1.155)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{S^1}\int_\Omega\big(\kappa(\vartheta)+\delta\vartheta^B+\delta\vartheta^{-1}\big)\frac{|\nabla\vartheta|^2}{\vartheta^2}\,\mathrm{d}x\,\mathrm{d}t + \int_{S^1}\int_{\partial\Omega}h(x,\vartheta)\Big(\vartheta+\frac{\Theta_0}{\vartheta}\Big)\,\mathrm{d}S\,\mathrm{d}t \\
+\varepsilon\delta\int_{S^1}\int_\Omega\Big(\frac{\Gamma\varrho^\Gamma}{\Gamma-1}+2\varrho^2\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{S^1}\int_\Omega\Big(\frac{\mathbb{S}(\vartheta,\nabla\mathbf{v}):\nabla\mathbf{v}}{\vartheta}+\frac{\delta}{\vartheta^2}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
+\varepsilon\delta\int_{S^1}\int_\Omega\frac{1}{\vartheta}(\Gamma\varrho^{\Gamma-2}+2)\,|\nabla\varrho|^2\,\mathrm{d}x\,\mathrm{d}t \le C\Big(1+\int_{S^1}\int_\Omega|\varrho\mathbf{g}\cdot\mathbf{v}|\,\mathrm{d}x\,\mathrm{d}t\Big). \quad (1.156)
\end{aligned}
$$

1.4.2 Better regularity of the pressure for $\varepsilon > 0$

In order to pass to the limit with $\varepsilon \to 0^+$, we have to establish better estimates of the density. Introducing the modified energy

$$
\begin{aligned}
\mathsf{E} = \sup_{t\in S^1}\mathcal{E}_\delta(t) = \sup_{t\in S^1}\int_\Omega\Big((\zeta+\varrho)\frac{|\mathbf{v}|^2}{2} + \zeta(\vartheta-\log\vartheta) \\
+H(\varrho,\vartheta) + \delta\Big(\frac{\varrho^\Gamma}{\Gamma-1}+\varrho^2\Big)\Big)\,\mathrm{d}x, \quad (1.157)
\end{aligned}
$$

where $H(\varrho,\vartheta) = \varrho\big(e(\varrho,\vartheta) - s(\varrho,\vartheta)\big)$ as before. We now aim at showing that

$$
\mathsf{E} + \int_{S^1}\int_\Omega\Big(\varrho^{\gamma+1}+\delta(\varrho^{\Gamma+1}+\varrho^3)\Big)\,\mathrm{d}x\,\mathrm{d}t \le C(\delta). \quad (1.158)
$$

Recalling (1.155) and (1.156) we obtain by already used mean value argument that

$$\mathsf{E} \le C\bigg(1 + \int_{S^1} \mathcal{E}_\delta(t)\,\mathrm{d}t + \int_{S^1}\int_\Omega |\varrho\mathbf{g}\cdot\mathbf{v}|\,\mathrm{d}x\,\mathrm{d}t + \varepsilon\int_{S^1}\int_\Omega \varrho s(\varrho,\vartheta)\,\mathrm{d}x\,\mathrm{d}t\bigg). \quad (1.159)$$

Due to the structure of E, we are able to push the last two terms on the right-hand side of (1.159) to its left-hand side. Thus, it remains to estimate the first integral, especially the terms with powers of density ϱ. For this purpose, we use a specific test function for the momentum equation (1.151)[18], namely ($M_0 = m|\Omega|$)

$$\mathbf{\Phi} = \mathcal{B}\left[\varrho - m\right],$$

yielding

$$\begin{aligned}
&\int_{S^1}\int_\Omega \big(p(\varrho,\vartheta) + \delta(\varrho^\Gamma + \varrho^2)\big)\varrho\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_{S^1}\int_\Omega \big(p(\varrho,\vartheta) + \delta(\varrho^\Gamma + \varrho^2)\big)m\,\mathrm{d}x\,\mathrm{d}t + \int_{S^1}\int_\Omega (\zeta + \varrho)\mathbf{v}\cdot\partial_t\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_{S^1}\int_\Omega (\varrho\mathbf{v}\otimes\mathbf{v}) : \nabla\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t + \int_{S^1}\int_\Omega \mathbb{S}(\vartheta,\nabla\mathbf{v}) : \nabla\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\quad + \int_{S^1}\int_\Omega \varrho\mathbf{g}\cdot\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t - \varepsilon\int_{S^1}\int_\Omega \nabla\varrho\cdot\nabla\mathbf{v}\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad + \frac{\varepsilon}{2}\int_{S^1}\int_\Omega (m-\varrho)\mathbf{v}\cdot\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t. \quad (1.160)
\end{aligned}$$

In order to estimate the right-hand side we proceed quite similarly as in the heuristic approach, the details can also be found in [44]. Consequently, we will consider here only the terms, which are different or more difficult. First, since we have for arbitrarily small $\eta > 0$ the bound

$$\int_{S^1}\int_\Omega \varrho(|\log\varrho| + |\log\vartheta|)\,\mathrm{d}x\,\mathrm{d}t \le C(\delta)(1 + \mathsf{E}^\eta), \quad (1.161)$$

we obtain using (1.29)

$$\begin{aligned}
&\int_{S^1}\int_\Omega (\varrho\mathbf{v}\otimes\mathbf{v}) : \nabla\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t \le C\,\|\mathbf{v}\|^2_{L^2(L^6)}\,\|\varrho\|^2_{L^\infty(L^3)} \\
&\qquad \le C\bigg(1 + \varepsilon\int_{S^1}\int_\Omega \varrho(|\log\varrho| + |\log\vartheta|)\,\mathrm{d}x\,\mathrm{d}t\bigg)^2 \mathsf{E}^{2/\Gamma} \le C(\delta)(1 + \mathsf{E}^{2/\Gamma+\eta}).
\end{aligned}$$

In order to deal with the term containing $\partial_t\mathbf{\Phi}$ we have to investigate first the continuity equation. Multiplying (1.150) by the density ϱ and integration over $S^1\times\Omega$ yields

$$\varepsilon\int_{S^1}\int_\Omega (|\nabla\varrho|^2 + \varrho^2)\,\mathrm{d}x\,\mathrm{d}t \le \varepsilon\int_{S^1}\int_\Omega m\varrho\,\mathrm{d}x\,\mathrm{d}t + \bigg|\int_{S^1}\int_\Omega \operatorname{div}(\varrho\mathbf{v})\varrho\,\mathrm{d}x\,\mathrm{d}t\bigg|. \quad (1.162)$$

[18]Recall that $\mathcal{B}$ stands for the Bogovskii operator from Theorem 4.13.

The last integral can be bounded after integration by parts as follows

$$\left|\int_{S^1}\int_\Omega \varrho^2 \operatorname{div}\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t\right| \le \|\nabla\mathbf{v}\|^2_{L^2(L^2)}\|\varrho\|^2_{L^\infty(L^4)}$$
$$\le C\frac{1}{\sqrt{\delta}}(1+\mathsf{E}^\eta)\Big(\sup_{t\in S^1}\int_\Omega \varrho^4\,\mathrm{d}x\Big)^{\frac{1}{2}} \le C\frac{1}{\sqrt{\delta}}(1+\mathsf{E}^{2/\Gamma+\eta}).$$

Thus,

$$\varepsilon\|\nabla\varrho\|^2_{L^2(L^2)} \le C(1+\delta^{-1/2}\mathsf{E}^{2/\Gamma+\eta}) \tag{1.163}$$
$$\varepsilon\|\nabla\varrho\|_{L^2(L^2)} \le C\sqrt{\varepsilon}(1+\delta^{-1/4}\mathsf{E}^{1/\Gamma+\eta}). \tag{1.164}$$

Further, noting that

$$\operatorname{div}\partial_t\mathbf{\Phi} = \partial_t\varrho = \varepsilon\Delta\varrho - \operatorname{div}(\varrho\mathbf{v}) + \varepsilon(m-\varrho)$$

we decompose $\partial_t\mathbf{\Phi} = \partial_t\mathbf{\Phi}^1 + \partial_t\mathbf{\Phi}^2$, where

$$\operatorname{div}\partial_t\mathbf{\Phi}^1 = \varepsilon\Delta\varrho \text{ in } \Omega,$$
$$\partial_t\mathbf{\Phi}^1 = 0 \text{ on } \partial\Omega.$$

Thus, we have from (1.164)

$$\left\|\partial_t\mathbf{\Phi}^1\right\|_{L^2(L^2)} \le C\varepsilon\|\nabla\varrho\|_{L^2(L^2)} \le C(\delta)\sqrt{\varepsilon}(1+\mathsf{E}^{1/\Gamma+\eta}).$$

Therefore,

$$\int_{S^1}\int_\Omega(\zeta+\varrho)\mathbf{v}\cdot\partial_t\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t = \int_{S^1}\int_\Omega(\zeta+\varrho)\mathbf{v}\cdot(\partial_t\mathbf{\Phi}^2+\partial_t\mathbf{\Phi}^1)\,\mathrm{d}x\,\mathrm{d}t$$
$$\le C(1+\mathsf{E}^{2/\Gamma+\eta}) + C(\delta)\sqrt{\varepsilon}\|\mathbf{v}\|_{L^2(L^6)}\|\varrho\|_{L^\infty(L^3)}(1+\mathsf{E}^{1/\Gamma+\eta})$$
$$\le C(\delta)(1+\mathsf{E}^{1/\Gamma+\eta})^2.$$

The integral $\varepsilon\int_{S^1}\int_\Omega \nabla\varrho\cdot\nabla\mathbf{v}\cdot\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t$ can be treated similarly as above. Note that there is again no difficulty concerning the term with ϑ^4, since we have from (1.156)

$$\|\vartheta\|_{L^3(L^9)} \le C\Big(1+\|\varrho\|^2_{L^2(L^{6/5})} + \varepsilon\int_{S^1}\int_\Omega \varrho s(\varrho,\vartheta)\,\mathrm{d}x\,\mathrm{d}t\Big) \le C(\delta)(1+\mathsf{E}^{2/\Gamma}).$$

As the structure of $\mathcal{E}_\delta$ gives us

$$\int_{S^1}\mathcal{E}_\delta(t)\,\mathrm{d}t \le C(1+\mathsf{E}^\beta),$$

with $\beta<1$, provided Γ is sufficiently large, we finally obtain (1.158). To summarize, we get the following a priori bounds independent of ε

Hence, we have shown

$$\sup_{t\in S^1}\int_\Omega\Big((\zeta+\varrho)\frac{|\mathbf{v}|^2}{2} + \zeta(\vartheta-\log\vartheta) + H(\varrho,\vartheta) + \delta\Big(\frac{\varrho^\Gamma}{\Gamma-1}+\varrho^2\Big)\Big)\,\mathrm{d}x$$
$$+\int_{S^1}\int_\Omega\big(\kappa(\vartheta)+\delta\vartheta^B+\delta\vartheta^{-1}\big)\frac{|\nabla\vartheta|^2}{\vartheta^2}\,\mathrm{d}x\,\mathrm{d}t + \int_{S^1}\int_{\partial\Omega}h(x,\vartheta)\left(\vartheta+\frac{\Theta_0}{\vartheta}\right)\mathrm{d}S\,\mathrm{d}t$$
$$+\int_{S^1}\int_\Omega\Big(\frac{\mathbb{S}(\vartheta,\nabla\mathbf{v}):\nabla\mathbf{v}}{\vartheta}+\frac{\delta}{\vartheta^2}\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{S^1}\int_\Omega\varrho^{\gamma+1}\,\mathrm{d}x\,\mathrm{d}t \le C(\delta),$$

with C in particular independent of ε.

1.4.3 Limit $\varepsilon \to 0^+$

The limit passage for $\varepsilon \to 0^+$ uses almost the same arguments as the forthcoming limit for $\delta \to 0^+$, and except the absence of the strong convergence of the initial densities, and the nonlinear boundary condition for the temperature also the same as in [48, Section 3.6]. Therefore, we skip it here, and present only the result of this limit.

We obtain for any $\delta, \zeta > 0$ a solution $(\varrho_\delta, \mathbf{v}_\delta, \vartheta_\delta)$ satisfying the continuity equation in the renormalized sense

$$\int_{S^1}\int_\Omega \Big(b(\varrho_\delta)\frac{\partial \psi}{\partial t} + b(\varrho_\delta)\mathbf{v}_\delta \cdot \nabla\psi + \big(b(\varrho_\delta) - b'(\varrho_\delta)\varrho_\delta\big)\operatorname{div}\mathbf{v}_\delta\psi\Big)\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{1.165}$$

for any $b \in C^\infty[0,\infty), b' \in C_c^\infty[0,\infty)$, and any $\psi \in C^\infty(S^1 \times \overline{\Omega})$.

The momentum equation (1.151) is satisfied with $\varepsilon = 0$, id est for all $\boldsymbol{\varphi} \in C_0^\infty(S^1 \times \Omega; \mathbb{R}^3)$ we have

$$\begin{aligned}\int_{S^1}\int_\Omega \Big((\zeta + \varrho_\delta)\mathbf{v}_\delta \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} + (\varrho_\delta \mathbf{v}_\delta \otimes \mathbf{v}_\delta) : \nabla\boldsymbol{\varphi}\Big)\,\mathrm{d}x\,\mathrm{d}t&\\ + \int_{S^1}\int_\Omega \Big(\big(p(\varrho_\delta, \vartheta_\delta) + \delta(\varrho_\delta^\Gamma + \varrho_\delta^2)\big)\operatorname{div}\boldsymbol{\varphi}\Big)\,\mathrm{d}x\,\mathrm{d}t&\\ = \int_{S^1}\int_\Omega \big(\mathbb{S}(\vartheta_\delta, \nabla\mathbf{v}_\delta) : \nabla\boldsymbol{\varphi} - \varrho_\delta \mathbf{g}\cdot\boldsymbol{\varphi}\big)\,\mathrm{d}x\,\mathrm{d}t.&\end{aligned} \tag{1.166}$$

The entropy inequality has the form ($\psi \in C^\infty(S^1 \times \overline{\Omega})$, $\psi \geq 0$)

$$\begin{aligned}\int_{S^1}\int_\Omega \big(\zeta \log\vartheta_\delta + \varrho_\delta s(\varrho_\delta, \vartheta_\delta)\big)\frac{\partial \psi}{\partial t} + \varrho_\delta s(\varrho_\delta, \vartheta_\delta)\mathbf{v}_\delta \cdot \nabla\psi\,\mathrm{d}x\,\mathrm{d}t&\\ - \int_{S^1}\int_\Omega \Big(\big(\kappa(\vartheta_\delta) + \delta\vartheta_\delta^B + \delta\vartheta_\delta^{-1}\big)\frac{\nabla\vartheta_\delta}{\vartheta_\delta}\cdot\nabla\psi\Big)\,\mathrm{d}x\,\mathrm{d}t&\\ + \int_{S^1}\int_\Omega \Big(\frac{\mathbb{S}(\vartheta_\delta, \nabla\mathbf{v}_\delta) : \nabla\mathbf{v}_\delta}{\vartheta_\delta} + \big(\kappa(\vartheta_\delta) + \delta\vartheta_\delta^B + \delta\vartheta_\delta^{-1}\big)\frac{|\nabla\vartheta_\delta|^2}{\vartheta_\delta^2}\Big)\,\mathrm{d}x\,\psi\,\mathrm{d}t&\\ \leq \int_{S^1}\int_{\partial\Omega}\frac{h(\vartheta_\delta - \Theta_0)}{\vartheta_\delta}\psi\,\mathrm{d}S\,\mathrm{d}t - \delta\int_{S^1}\int_\Omega \frac{1}{\vartheta_\delta^2}\psi\,\mathrm{d}x\,\mathrm{d}t,&\end{aligned} \tag{1.167}$$

and the total energy balance ($\psi \in C^\infty(S^1)$)

$$\begin{aligned}\int_{S^1}\int_\Omega \Big((\zeta + \varrho_\delta)\frac{|\mathbf{v}_\delta|^2}{2} + \zeta\vartheta_\delta + \varrho_\delta e(\varrho_\delta, \vartheta_\delta) + \delta\Big(\frac{\varrho_\delta^\Gamma}{\Gamma - 1} + \varrho_\delta^2\Big)\Big)\,\mathrm{d}x\frac{\partial \psi}{\partial t}\,\mathrm{d}t&\\ = \int_{S^1}\int_{\partial\Omega} h(x, \vartheta_\delta)(\vartheta_\delta - \Theta_0)\,\mathrm{d}S\,\psi\,\mathrm{d}t - \int_{S^1}\int_\Omega \Big(\varrho_\delta \mathbf{g}\cdot\mathbf{v}_\delta + \frac{\delta}{\vartheta_\delta}\Big)\,\mathrm{d}x\,\psi\,\mathrm{d}t.&\end{aligned} \tag{1.168}$$

To simplify our further considerations, let us introduce a positive Radon measure σ_δ (slightly different from σ introduced in Proposition 1.11) satisfying for all

$\psi \in C^\infty(S^1 \times \overline{\Omega})$

$$\langle \sigma_\delta, \psi \rangle = -\int_{S^1}\int_\Omega \big(\zeta \log \vartheta_\delta + \varrho_\delta s(\varrho_\delta, \vartheta_\delta)\big)\frac{\partial \psi}{\partial t} + \varrho_\delta s(\varrho_\delta, \vartheta_\delta)\mathbf{v}_\delta \cdot \nabla \psi \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{S^1}\int_\Omega \Big(\big(\kappa(\vartheta_\delta) + \delta \vartheta_\delta^B + \delta \vartheta_\delta^{-1}\big)\frac{\nabla \vartheta_\delta}{\vartheta_\delta}\cdot \nabla \psi\Big)\,\mathrm{d}x\,\mathrm{d}t \tag{1.169}$$
$$+ \int_{S^1}\int_{\partial\Omega} h(x, \vartheta_\delta)\psi \,\mathrm{d}S\,\mathrm{d}t,$$

then formula (1.167) reads ($\psi \geq 0$)

$$\langle \sigma_\delta, \psi \rangle \geq \int_{S^1}\int_\Omega \frac{\mathbb{S}(\vartheta_\delta, \nabla \mathbf{v}_\delta) : \nabla \mathbf{v}_\delta}{\vartheta}\psi \,\mathrm{d}x\,\mathrm{d}t + \delta \int_{S^1}\int_\Omega \frac{1}{\vartheta_\delta^2}\psi \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{S^1}\int_{\partial\Omega} h(x, \vartheta_\delta)\frac{\Theta_0}{\vartheta_\delta}\psi \,\mathrm{d}S\,\mathrm{d}t + \int_{S^1}\int_\Omega \big(\kappa(\vartheta_\delta) + \delta\vartheta_\delta^B + \delta\vartheta_\delta^{-1}\big)\frac{|\nabla \vartheta_\delta|^2}{\vartheta_\delta^2}\psi \,\mathrm{d}x\,\mathrm{d}t. \tag{1.170}$$

Now, we are able to to set $\zeta = \delta$ and perform the last limit passage.

1.4.4 Limit $\delta \to 0^+$

The limit passage for $\delta \to 0^+$ is the crucial step in our considerations. First of all, we need to derive estimates independent of the approximative parameter δ. This will be done in the same manner as in the heuristic approach in Section 1.2; the only additional estimates which we need, are those dependent on δ. Combining (1.167) and (1.168) we get

$$\|\mathbf{v}_\delta\|^2_{L^2(W_0^{1,2}(\Omega))} + \Big\|\nabla(\vartheta_\delta^{3/2})\Big\|^2_{L^2(L^2)} + \|\nabla(\log \vartheta_\delta)\|^2_{L^2(L^2)}$$
$$+ \Big\|\frac{1}{\vartheta_\delta}\Big\|_{L^1(S^1\times\partial\Omega)} + \|\vartheta_\delta\|^2_{L^2(S^1\times\partial\Omega)} + \delta\Big\|\nabla(\vartheta_\delta^{B/2})\Big\|^2_{L^2(L^2)}$$
$$+ \delta \int_{S^1}\int_\Omega \frac{1}{\vartheta_\delta^2}\,\mathrm{d}x\,\mathrm{d}t \leq C\Big(1 + \|\varrho_\delta\|^{6/5}_{L^2(L^{6/5})}\Big). \tag{1.171}$$

Further, we can deduce for the (modified) total energy

$$\mathcal{E}_\delta(t) = \int_\Omega \bigg(\varrho_\delta \frac{|\mathbf{v}_\delta|^2}{2} + \varrho_\delta e(\varrho_\delta, \vartheta_\delta) + \delta\Big(\frac{|\mathbf{v}_\delta|^2}{2} + \frac{\vartheta_\delta + |\log \vartheta_\delta|}{2} + \frac{\varrho_\delta^\Gamma}{\Gamma - 1} + \varrho_\delta^2\Big)\bigg)\,\mathrm{d}x$$

that

$$\sup_{t\in S^1}\mathcal{E}_\delta(t) \leq C\bigg(1 + \int_{S^1}\int_\Omega \varrho_\delta^\gamma + \delta\left(\varrho_\delta^2 + \varrho_\delta^\Gamma\right)\mathrm{d}x\,\mathrm{d}t\bigg). \tag{1.172}$$

Following closely Section 1.2.2 we finally obtain the estimates of the form

$$\sup_{t\in S^1}\mathcal{E}_\delta(t) + \int_{S^1}\int_\Omega \Big(\varrho_\delta^{\gamma a} + \delta\big(\varrho_\delta^{2+\gamma(a-1)} + \varrho_\delta^{\Gamma+\gamma(a-1)}\big)\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \|\vartheta_\delta\|_{L^{13/3}(S^1\times\partial\Omega)} + \|\mathbf{v}_\delta\|^2_{L^2(W_0^{1,2}(\Omega))} + \Big\|\nabla(\vartheta_\delta^{3/2})\Big\|^2_{L^2(L^2)}$$
$$+ \|\nabla(\log \vartheta_\delta)\|^2_{L^2(L^2)} + \delta\Big\|\nabla(\vartheta_\delta^{B/2})\Big\|^2_{L^2(L^2)} \leq C \tag{1.173}$$

with $a > 1$ (see (1.89)), and C independent of the approximation parameters.

From these estimates we obtain the quadruple $(\varrho, \mathbf{v}, \vartheta, \sigma)$ such that

$$
\begin{aligned}
&\varrho_\delta \rightharpoonup^* \varrho \text{ in } L^\infty\left(S^1; L^\gamma(\Omega)\right)) \text{ and in } L^p(S^1 \times \Omega) \text{ for some } \; p > 1, \\
&\mathbf{v}_\delta \rightharpoonup \mathbf{v} \text{ in } L^2(S^1; W^{1,2}(\Omega)) \hookrightarrow L^2\left(S^1; L^6(\Omega)\right), \\
&\vartheta_\delta \rightharpoonup^* \vartheta \text{ in } L^\infty\left(S^1; L^4(\Omega)\right), \text{ and in } L^2\big(S^1; W^{1,2}(\Omega)\big), \\
&\sigma_\delta \rightharpoonup^* \sigma \text{ in } \mathcal{M}(S^1 \times \overline{\Omega}).
\end{aligned}
$$

Additionally, we deduce by the Arzelà–Ascoli type of argument from Theorem 4.5 that

$$
\begin{aligned}
&\varrho_\delta \to \varrho \text{ in } C_{\text{weak}}(S^1, L^\gamma(\Omega)), \\
&\varrho_\delta \mathbf{v}_\delta \to \varrho \mathbf{v} \text{ in } C_{\text{weak}}(S^1, L^{\frac{2\gamma}{\gamma+1}}(\Omega)), \\
&b(\varrho_\delta) \to \overline{b(\varrho)} \text{ in } C_{\text{weak}}(S^1, L^p(\Omega)).
\end{aligned}
\tag{1.174}
$$

Here, and in what follows, the bar over a nonlinear expression denotes its weak limit. The main difficulty will be to show that we have for certain nonlinear g's $g(\varrho, \mathbf{v}, \vartheta) = \overline{g(\varrho, \mathbf{v}, \vartheta)}$. Therefore, we need tools of the theory of compensated compactness from Sections 4.4–4.6.

Strong convergence of temperature

First, let us apply Div-Curl Lemma 4.20 to the following four-dimensional vector fields

$$
\begin{aligned}
\mathbf{V}_\delta &:= \left[\delta \log \vartheta_\delta + \varrho_\delta s(\varrho_\delta, \vartheta_\delta),\; \varrho_\delta s(\varrho_\delta, \vartheta_\delta)\mathbf{v}_\delta - \left(\Big(\kappa(\vartheta_\delta) + \frac{\delta}{\vartheta} + \delta \vartheta^B\Big)\frac{\nabla \vartheta_\delta}{\vartheta_\delta}\right)\right] \\
\mathbf{U}_\delta &:= \big[T_k(\vartheta_\delta),\; 0,\; 0,\; 0\big],
\end{aligned}
$$

where we have introduced a concave smooth cut-off function

$$
T_k(z) = kT\left(\frac{z}{k}\right), \quad T(z) = \begin{cases} z \text{ for } z \in [0,1] \\ 2 \text{ for } z \in [3, \infty). \end{cases}
$$

The structure of $s(\varrho, \vartheta)$ together with the estimates (1.173) ensures that $\mathbf{V}_\delta$ is uniformly bounded in $L^p(S^1 \times \Omega)$ for some $p > 1$. In addition, we observe that the terms in the entropy inequality (1.167) with δ vanish as $\delta \to 0$ in the sense of weak convergence in $L^p(S^1 \times \Omega)$ $(p > 1)$. Further, (1.167) and the estimates below implies that all assumptions of Lemma 4.20 are satisfied for $\mathbf{V}_\delta$, $\mathbf{U}_\delta$, hence

$$
\overline{T_k(\vartheta)\varrho s_0(\varrho, \vartheta)} + \frac{4a}{3}\overline{T_k(\vartheta)\vartheta^3} = \overline{T_k(\vartheta)}\;\overline{\varrho s_0(\varrho, \vartheta)} + \frac{4a}{3}\overline{T_k(\vartheta)}\;\overline{\vartheta^3}. \tag{1.175}
$$

Note that we tacitly extended the function $\varrho s_0(\varrho, \vartheta)$ continuously to $\varrho = 0$. Now, we would like to deduce in the same manner as in [44] that

$$
\overline{T_k(\vartheta)\varrho s_0(\varrho, \vartheta)} \geq \overline{T_k(\vartheta)}\;\overline{\varrho s_0(\varrho, \vartheta)}. \tag{1.176}
$$

This is indeed true since according to the fact that $\vartheta \mapsto s_0(\varrho, \vartheta)$ is monotone, we have

$$
\left(\varrho_\delta s_0\big(\varrho_\delta, T_k(\vartheta_\delta)\big) - \varrho_\delta s_0\big(\varrho_\delta, \overline{T_k(\vartheta)}\big)\right)\left(T_k(\vartheta_\delta) - \overline{T_k(\vartheta)}\right) \geq 0,
$$

and as a consequence of De la Vallée-Poussin criterion, see [48, Corollary 10.2], also

$$\sup_{\delta>0} \left\| \varrho_\delta s_0(\varrho_\delta, \vartheta_\delta) T_k(\vartheta_\delta) - \varrho_\delta s_0\big(\varrho_\delta, T_k(\vartheta_\delta)\big) T_k(\vartheta_\delta) \right\|_{L^1(S^1\times\Omega)} \to 0,$$
$$\sup_{\delta>0} \left\| \varrho_\delta s_0(\varrho_\delta, \vartheta_\delta) \overline{T_k(\vartheta)} - \varrho_\delta s_0\big(\varrho_\delta, T_k(\vartheta_\delta)\big) \overline{T_k(\vartheta)} \right\|_{L^1(S^1\times\Omega)} \to 0$$

for $k \to \infty$. Thus, as a matter of fact, in order to prove (1.176), it is enough to show that

$$\varrho_\delta s_0\big(\varrho_\delta, \overline{T_k(\vartheta)}\big)\big(T_k(\vartheta_\delta) - \overline{T_k(\vartheta)}\big) \to 0 \text{ in } L^1_{\text{loc}}(S^1 \times \Omega). \tag{1.177}$$

Due to (1.27)–(1.28) we have

$$\left| \varrho_\delta s_0\big(\varrho_\delta, \overline{T_k(\vartheta)}\big) - \varrho_\delta s_0\big(\varrho_\delta, \Theta\big) \right| \le C \varrho_\delta \left| \log\big(\overline{T_k(\vartheta)} - \log \Theta\big) \right|,$$

hence (1.177) follows from the fact that logarithm is concave, and hence

$$\overline{\log\big(T_k(\vartheta)\big)} \le \log\big(\overline{T_k(\vartheta)}\big),$$

see a counterpart of Theorem 4.18. Therefore we can get by combining relations (1.175) and (1.176)

$$\overline{T_k(\vartheta)\vartheta^3} \le \overline{T_k(\vartheta)}\, \overline{\vartheta^3}. \tag{1.178}$$

Having (1.178) in hand we can apply Lemma 4.17, since $Q(z) = z^3$ is strictly increasing, and we obtain that

$$\overline{T_k(\vartheta)\vartheta^3} = \overline{T_k(\vartheta)}\, \overline{\vartheta^3}, \tag{1.179}$$

hence consequently, since ϑ is bounded in $L^{17/3}(S^1 \times \Omega)$, also

$$\overline{\vartheta^4} = \overline{\vartheta^3}\, \vartheta,$$

from where we finally conclude

$$\vartheta_\delta \to \vartheta \text{ a.e. in } S^1 \times \Omega. \tag{1.180}$$

Concerning the nonlinear boundary term $h(x,\vartheta)\vartheta$ we have compactness according to the key estimate (1.91)

$$\|\vartheta\|^{3/2}_{L^{13/3}(\partial\Omega)} \le C \left\| \vartheta^{3/2} \right\|_{W^{1,2}(\Omega)}. \tag{1.181}$$

Hence, due to the standard interpolation argument we can conclude the strong convergence in $L^p(S^1 \times \partial\Omega)$ for all $p < \frac{13}{3}$. On the other hand, the term $h(x,\vartheta)/\vartheta$ is bounded merely in $L^1(S^1 \times \partial\Omega)$ uniformly with respect to δ, so we have to identify the weak limit as a part of the non-negative measure σ using its convexity and Theorem 4.18.

The last step is to show the pointwise convergence of densities in order to identify the limit in the pressure. For this purpose we will use nowadays "classical" arguments exploited by Lions [85] and Feireisl [37] including the effective viscous flux identity, commutator lemma for the Riesz operators, oscillations defect measure or the limit renormalized continuity equation. Although we use simply the same arguments as in [44], we present this part here to make the limit passage in this section as self-contained as possible.

Effective viscous flux identity

In order to get the weak compactness identity for effective viscous flux, we subtract two identities. The first one is the limit momentum equation tested by $\boldsymbol{\varphi} = \psi\nabla\Delta^{-1}[T_k(\varrho_\delta)\chi_\Omega]$.[19] The second one is obtained by testing the momentum equation (1.166) by $\boldsymbol{\varphi} = \psi\nabla\Delta^{-1}[\overline{T_k(\varrho_\delta)}\chi_\Omega]$, and then taking the limit as $\delta \to 0$; in both cases $\psi \in C_c^\infty(S^1 \times \Omega)$ is an arbitrary cut-off function. Recalling the notation for Riesz transform (4.108)

$$\mathcal{R}_{ij}[v] = \mathcal{F}^{-1}\left[\frac{\xi_i\xi_j}{|\xi|^2}\mathcal{F}(v)(\xi)\right] \tag{1.182}$$

and dropping the terms which clearly converge to their analogues we obtain

$$\begin{aligned}
&\lim_{\delta\to 0+}\int_{S^1}\int_\Omega \psi(t,x)\Big(p(\varrho_\delta,\vartheta_\delta)T_k(\varrho_\delta) - \mathbb{S}(\vartheta_\delta,\mathbf{v}_\delta) : \mathcal{R}[T_k(\varrho_\delta)\chi_\Omega]\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_{S^1}\int_\Omega \psi(t,x)\left(\overline{p(\varrho,\vartheta)}\;\overline{T_k(\varrho)} - \mathbb{S}(\vartheta,\mathbf{v}) : \mathcal{R}[\overline{T_k(\varrho)}\chi_\Omega]\right)\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \lim_{\delta\to 0+}\int_{S^1}\int_\Omega \psi(t,x)\Big(T_k(\varrho_\delta)\mathbf{v}_\delta\cdot\mathcal{R}[\varrho_\delta\mathbf{v}_\delta\chi_\Omega] \\
&\qquad\qquad - \varrho_\delta(\mathbf{v}_\delta\otimes\mathbf{v}_\delta) : \mathcal{R}[T_k(\varrho_\delta)\chi_\Omega]\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int_{S^1}\int_\Omega \psi(t,x)\Big(\overline{T_k(\varrho)}\mathbf{v}\cdot\mathcal{R}[\varrho\mathbf{v}\chi_\Omega] - \varrho(\mathbf{v}\otimes\mathbf{v}) : \mathcal{R}[\overline{T_k(\varrho)}\chi_\Omega]\Big)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{1.183}$$

Now, we will use two commutators lemmata from Section 4.5. The convergences (1.174) imply

$$\begin{aligned}
T_k(\varrho_\delta) &\to \overline{T_k(\varrho_\delta)} \text{ in } C_{\text{weak}}(S^1, L^q(\Omega)),\ \text{for } 1 \le q < \infty, \\
\varrho_\delta\mathbf{v}_\delta &\to \varrho\mathbf{v} \text{ in } C_{\text{weak}}(S^1, L^{\frac{2\gamma}{\gamma+1}}(\Omega)),
\end{aligned}$$

thus Lemma 4.21 yields

$$\begin{aligned}
&\int_{S^1}\int_\Omega \psi(t,x)\mathbf{v}_\delta\cdot\Big(T_k(\varrho_\delta)\mathcal{R}[\varrho_\delta\mathbf{v}_\delta\chi_\Omega] - \varrho_\delta\mathcal{R}[T_k(\varrho_\delta)\chi_\Omega]\mathbf{v}_\delta\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \to \int_{S^1}\int_\Omega \psi(t,x)\mathbf{v}\cdot\Big(\overline{T_k(\varrho)}\mathcal{R}[\varrho\mathbf{v}\chi_\Omega] - \varrho\mathcal{R}[\overline{T_k(\varrho)}\chi_\Omega]\mathbf{v}\Big)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{1.184}$$

hence combining it with (1.183)

$$\begin{aligned}
&\int_{S^1}\int_\Omega \psi(t,x)\left(\overline{p(\varrho,\vartheta)T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\;\overline{T_k(\varrho)}\right)\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_{S^1}\int_\Omega \psi(t,x)\left(\overline{\mathbb{S}(\vartheta,\mathbf{v}) : \mathcal{R}[T_k(\varrho)\chi_\Omega]} - \mathbb{S}(\vartheta,\mathbf{v}) : \mathcal{R}[\overline{T_k(\varrho)}\chi_\Omega]\right)\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{1.185}$$

Further, denoting

$$\omega(\vartheta,\mathbf{v}) = \Big(\mathcal{R} : \big[\psi(t,x)\mu(\vartheta)\big(\nabla\mathbf{v}+\nabla\mathbf{v}^T\big)\big] - \psi(t,x)\mu(\vartheta)\mathcal{R} : \big[\nabla\mathbf{v}+\nabla\mathbf{v}^T\big]\Big),$$

[19] Here and in what follows χ_Ω denotes the characteristic function of the set Ω, the operator of inverse divergence $\nabla\Delta^{-1}$ is defined through (4.107).

we get from the basic properties of the Riesz transform

$$
\begin{aligned}
\int_{S^1}\int_\Omega \psi(t,x)\overline{\mathbb{S}(\vartheta,\mathbf{v}) : \mathcal{R}[T_k(\varrho)\chi_\Omega]}\,\mathrm{d}x\,\mathrm{d}t &= \lim_{\delta\to0+}\int_{S^1}\int_\Omega T_k(\varrho_\delta)\,\omega(\vartheta_\delta,\mathbf{v}_\delta)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \lim_{\delta\to0+}\int_{S^1}\int_\Omega \psi(t,x)\left(\frac{4}{3}\mu(\vartheta_\delta)+\eta(\vartheta_\delta)\right)\operatorname{div}\mathbf{v}_\delta\,T_k(\varrho_\delta)\,\mathrm{d}x\,\mathrm{d}t, \quad (1.186)
\end{aligned}
$$

as well as

$$
\begin{aligned}
\int_{S^1}\int_\Omega \psi(t,x)\mathbb{S}(\vartheta,\mathbf{v}) : \mathcal{R}[\overline{T_k(\varrho)}\chi_\Omega]\,\mathrm{d}x\,\mathrm{d}t &= \int_{S^1}\int_\Omega \overline{T_k(\varrho)}\,\omega(\vartheta,\mathbf{v})\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_{S^1}\int_\Omega \psi(t,x)\left(\frac{4}{3}\mu(\vartheta)+\eta(\vartheta)\right)\operatorname{div}\mathbf{v}\,\overline{T_k(\varrho)}\,\mathrm{d}x\,\mathrm{d}t. \quad (1.187)
\end{aligned}
$$

According to Lemma 4.22, the vector fields

$$
\mathbf{V}_\delta := [T_k(\varrho_\delta), T_k(\varrho_\delta)\mathbf{v}_\delta], \text{ and } \mathbf{U}_\delta := [\omega(\vartheta_\delta,\mathbf{v}_\delta),\ 0,\ 0,\ 0]
$$

satisfy the hypotheses of Lemma 4.20, hence we obtain

$$
\overline{T_k(\varrho_\delta)\omega(\vartheta,\mathbf{v})} = \overline{T_k(\varrho_\delta)}\,\omega(\vartheta,\mathbf{v}). \tag{1.188}
$$

From (1.185)–(1.187), and (1.188) we finally get the famous effective viscous flux identity

$$
\left(\frac{4}{3}\mu(\vartheta)+\eta(\vartheta)\right)\left(\overline{T_k(\varrho)\operatorname{div}\mathbf{v}} - \overline{T_k(\varrho)}\operatorname{div}\mathbf{v}\right) = \overline{p_0(\varrho,\vartheta)T_k(\varrho)} - \overline{p_0(\varrho,\vartheta)}\,\overline{T_k(\varrho)}. \tag{1.189}
$$

Oscillations defect measure and limit renormalized continuity equation

Now, we aim at applying Lemma 4.25 in order to show that the limit continuity equation is satisfied also in the renormalized sense. As a matter of fact, since $\nabla\mathbf{v}_\delta \rightharpoonup \nabla\mathbf{v}$ weakly in $L^2(S^1\times\Omega)$, we need to prove that oscillations defect measure defined by (1.193)

$$
\mathbf{osc_q}[\varrho_\delta\to\varrho](S^1\times\Omega) := \sup_{k>0}\limsup_{\delta\to0+}\int_{S^1}\int_\Omega |T_k(\varrho_\delta)-T_k(\varrho)|^q\,\mathrm{d}x\,\mathrm{d}t
$$

is bounded for some $q>2$. We estimate for any $\psi\in C_c^\infty(S^1\times\Omega)$, $\psi\geq0$

$$
\begin{aligned}
\limsup_{\delta\to0+}&\int_{S^1}\int_\Omega \psi(t,x)\,|T_k(\varrho_\delta)-T_k(\varrho)|^{\gamma+1}\,\mathrm{d}x\,\mathrm{d}t \\
&\leq \limsup_{\delta\to0+}\int_{S^1}\int_\Omega \psi(t,x)\big(T_k(\varrho_\delta)-T_k(\varrho)\big)\big(\varrho_\delta^\gamma-\varrho^\gamma\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\leq \int_{S^1}\int_\Omega \psi(t,x)\left(\overline{\varrho^\gamma T_k(\varrho)} - \overline{\varrho^\gamma}\,\overline{T_k(\varrho)}\right)\mathrm{d}x\,\mathrm{d}t \\
&\leq \int_{S^1}\int_\Omega \psi(t,x)\left(\overline{p_0(\varrho,\vartheta)T_k(\varrho)} - \overline{p_0(\varrho,\vartheta)}\,\overline{T_k(\varrho)}\right)\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
\tag{1.190}
$$

where we have used (1.26) and also that according to the fact that $z\mapsto z^\gamma$ is convex and $T_k(z)$ is concave, we have

$$
\big(\overline{\varrho^\gamma}-\varrho^\gamma\big)\big(T_k(\varrho)-\overline{T_k(\varrho)}\,\big)\geq0,
$$

see e.g. [115, Lemma 18].

Next, the right-hand side of inequality (1.190) will be estimated by means of (1.189). We denote $G_k(t,x,z) = \big|T_k(z) - T_k\big(\varrho(t,x)\big)\big|^{\gamma+1}$, so that for any k

$$\overline{G_k(\cdot,\cdot,\varrho)} \le \overline{p_0(\varrho,\vartheta)T_k(\varrho)} - \overline{p_0(\varrho,\vartheta)}\;\overline{T_k(\varrho)}$$

and from the effective viscous flux identity (1.189) also

$$\overline{G_k(\cdot,\cdot,\varrho)} \le \Big(\frac{4}{3}\mu(\vartheta) + \eta(\vartheta)\Big)\Big(\overline{T_k(\varrho)\operatorname{div}\mathbf{v}} - \overline{T_k(\varrho)}\operatorname{div}\mathbf{v}\Big).$$

Therefore,

$$\begin{aligned}
&\int_{S^1}\int_\Omega \frac{1}{1+\vartheta}\,\overline{G_k(t,x,\varrho)}\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\le C\sup_{\delta>0}\|\operatorname{div}\mathbf{v}_\delta\|_{L^2(S^1\times\Omega)}\limsup_{\delta\to0+}\|T_k(\varrho_\delta)-T_k(\varrho)\|_{L^2(S^1\times\Omega)}\\
&\qquad\le C\limsup_{\delta\to0+}\|T_k(\varrho_\delta)-T_k(\varrho)\|_{L^2(S^1\times\Omega)}\,.
\end{aligned}\tag{1.191}$$

On the other hand,

$$\begin{aligned}
&\int_{S^1}\int_\Omega |T_k(\varrho_\delta)-T_k(\varrho)|^{\frac{17(\gamma+1)}{20}}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le C\left(\int_{S^1}\int_\Omega \frac{1}{1+\vartheta}|T_k(\varrho_\delta)-T_k(\varrho)|^{\gamma+1}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{17}{20}}\|1+\vartheta\|^{\frac{17}{20}}_{L^{\frac{17}{3}}(S^1\times\Omega)}\,,
\end{aligned}\tag{1.192}$$

where $\frac{17(\gamma+1)}{20} > 2$, even for any $\gamma > \frac{23}{17}$, which is in our case undoubtedly satisfied, hence we got[20]

$$\mathbf{osc}_{\boldsymbol{q}}[\varrho_\delta\to\varrho](S^1\times\Omega) \le C \tag{1.193}$$

with certain $q > 2$. Having (1.193) in hands, Lemma 4.25 yields the satisfaction of the limit renormalized continuity equation in a straightforward way. Particularly, by setting

$$b_k(\varrho) = \varrho\int_1^\varrho \frac{T_k(z)}{z^2}\mathrm{d}z,$$

in the renormalized continuity equation and its limit version we deduce[21]

$$\lim_{\delta\to0}\int_{S^1}\int_\Omega T_k(\varrho_\delta)\operatorname{div}\mathbf{v}_\delta\,\mathrm{d}x\,\mathrm{d}t = \int_{S^1}\int_\Omega \overline{T_k(\varrho)}\operatorname{div}\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{1.194}$$

Putting (1.189) and (1.194) together we get

$$\lim_{k\to\infty}\int_{S^1}\int_\Omega \frac{\overline{p_0(\varrho,\vartheta)T_k(\varrho)} - \overline{p_0(\varrho,\vartheta)}\;\overline{T_k(\varrho)}}{\frac{4}{3}\mu(\vartheta)+\eta(\vartheta)}\,\mathrm{d}x\,\mathrm{d}t = 0,$$

and

$$\lim_{k\to\infty}\int_{S^1}\int_\Omega \frac{\overline{\varrho^\gamma T_k(\varrho)} - \overline{\varrho^\gamma}\;\overline{T_k(\varrho)}}{\frac{4}{3}\mu(\vartheta)+\eta(\vartheta)}\,\mathrm{d}x\,\mathrm{d}t = 0.$$

[20] In the proof of Theorem 1.3, we control $\vartheta \in L^{\frac{41}{9}}(S^1\times\Omega)$, hence we get the same conclusion with $q = \frac{533}{250} > 2$.

[21] Note $b_k(\varrho) - b_k'(\varrho)\varrho = -T_k(\varrho)$.

Next, mimicking the estimates (1.190) we obtain

$$\lim_{k\to\infty} \limsup_{\delta\to 0+} \int_{S^1} \int_\Omega \frac{1}{1+\vartheta} |T_k(\varrho_\delta) - T_k(\varrho)|^{\gamma+1} \,\mathrm{d}x\,\mathrm{d}t = 0,$$

which combined with (1.192) implies

$$\lim_{k\to\infty} \limsup_{\delta\to 0+} \int_{S^1} \int_\Omega |T_k(\varrho_\delta) - T_k(\varrho)|^{q} \,\mathrm{d}x\,\mathrm{d}t = 0.$$

Thus, writing

$$\begin{aligned} \|\varrho_\delta - \varrho\|_{L^1(S^1\times\Omega)} \leq \|\varrho_\delta - T_k(\varrho_\delta)\|_{L^1(S^1\times\Omega)} + \|T_k(\varrho_\delta) - T_k(\varrho)\|_{L^1(S^1\times\Omega)} \\ + \|T_k(\varrho) - \varrho\|_{L^1(S^1\times\Omega)} \quad \to \quad 0 \end{aligned}$$

we have proved the desired conclusion

$$\varrho_\delta \to \varrho \text{ a.e. in } S^1 \times \Omega.$$

This completes the proof of Theorem 1.2.

2. Steady solutions to the Navier–Stokes–Allen–Cahn system

In this chapter we study the existence of steady weak solutions to a model for two-constituent compressible isothermal flow. The model under consideration is a variant of a model proposed by Blesgen [10], see also Heida, Málek and Rajagopal [60]. We set in the model from the introductory part $\nu(\varrho) = 1$, $C_0 = 1$, $\vartheta = const.$, and we obtain the following system of partial differential equations

$$\operatorname{div}(\varrho\mathbf{v}) = 0, \tag{2.1}$$

$$\operatorname{div}(\varrho\mathbf{v}\otimes\mathbf{v}) = \operatorname{div}\mathbb{T} + \varrho\mathbf{g}, \tag{2.2}$$

$$\operatorname{div}(\varrho c\mathbf{v}) = \overset{+}{c}, \tag{2.3}$$

$$\varrho\overset{+}{c} = \Delta c - \varrho\frac{\partial f}{\partial c}(\varrho, c), \tag{2.4}$$

where the corresponding stress tensor $\mathbb{T}$ is given by

$$\mathbb{T}(\nabla\mathbf{v}, \nabla c, \varrho, c) = 2\mu\mathbb{D}(\mathbf{v}) + \lambda\operatorname{div}\mathbf{v}\mathbb{I} - \Big(\nabla c\otimes\nabla c - \frac{|\nabla c|^2}{2}\mathbb{I}\Big) - p(\varrho, c)\mathbb{I}.$$

We suppose the viscosity coefficients[1] $\mu > 0$ and $2\mu + 3\lambda > 0$ to be constant. The free energy contains the energy of mixing, see e.g. [81], in the form of the so-called logarithmic potential[2]

$$f(\varrho, c) = \frac{1}{\gamma - 1}\varrho^{\gamma-1} + \big(a_1 c + a_2(1-c)\big)\log\varrho + c\log c + (1-c)\log(1-c) + b(c)$$

with some $\gamma > 1$[3], $a_1, a_2 \geq 0$ and b a smooth bounded function with $|b'(c)| \leq C$. Moreover we assume without loss of generality that $a_1 \geq a_2$, and we denote for the sake of simplicity $a = a_1 - a_2$, $d = a_1$, and $L(c) = c\log c + (1-c)\log(1-c)$. The logarithmic terms related to the entropy of the system assure that $c \in [0, 1]$ almost everywhere, since

$$\begin{aligned}\frac{\partial f}{\partial c}(\varrho, c) =& \log c - \log(1-c) + (a_1 - a_2)\log\varrho + b'(c)\\ =& L'(c) + a\log\varrho + b'(c).\end{aligned}$$

Consequently, the pressure $p = \varrho^2\dfrac{\partial f}{\partial\varrho}$ takes the form $p(\varrho, c) = \varrho^\gamma + \varrho\big(ac + d\big)$.

[1] Note that we use here slightly different notation for the viscosity coefficients when compared to Chapter 1.

[2] We could assume also more general pressure law function similarly to Chapter 1, but it would lead only to additional unnecessary technicalities, so we omit it. Similarly, we could consider more general singular functions $L(c)$.

[3] The values of γ which ensure the existence result will be specified later.

The fluid is contained in a smooth bounded domain $\Omega \subset \mathbb{R}^3$, we supply the equations in the domain with boundary conditions

$$\mathbf{v}\cdot\mathbf{n} = 0, \tag{2.5}$$
$$\mathbf{n}\cdot\mathbb{T}\cdot\boldsymbol{\tau}^n + \alpha\mathbf{v}\cdot\boldsymbol{\tau}^n = 0, \tag{2.6}$$
$$\nabla c\cdot\mathbf{n} = 0 \text{ at } \partial\Omega, \tag{2.7}$$

where parameter $\alpha > 0$ represents the friction on the boundary[4], $\boldsymbol{\tau}^n$, $n = 1,2$ are two linearly independent tangent vectors to $\partial\Omega$, and $\mathbf{n}$ denotes the normal vector.

The solutions are parametrized by means of the condition

$$\int_\Omega \varrho \,\mathrm{d}x = M_0. \tag{2.8}$$

The fluid is driven by an external force $\mathbf{g} \in L^\infty(\Omega, \mathbb{R}^3)$.

2.1 Definition of the solutions, the main result

We aim at constructing weak solutions to the Navier–Stokes–Allen–Cahn system. We use a technique developed by Mucha and Pokorný [100, 132], which was modified for the Navier–Stokes–Fourier system [101, 122], as well. Their method allows to obtain solutions with pointwise bounded densities, which seems to be the best possible regularity for weak solutions with the presence of vacuum, see Lions [85, Example 6.4]. We introduce the following definition of weak solutions to system (2.1)–(2.4).

Definition 2.1. Let $M_0 > 0$ be a given constant, $\gamma > 3$, we say that quadruple $\varrho, \mathbf{v}, \overset{+}{c}, c$ is a weak solution to the steady Navier–Stokes–Allen–Cahn system, if $\varrho \in L^\gamma(\Omega)$, $\varrho \geq 0$ a.e. in Ω, with $\int_\Omega \varrho \,\mathrm{d}x = M_0$, $\mathbf{v} \in W^{1,2}(\Omega)$, $\overset{+}{c} \in L^2(\Omega)$, $c \in W^{1,2}(\Omega)$, $0 \leq c \leq 1$ a.e. on $\{\varrho > 0\}$, with $\varrho L'(c) \in L^{2\gamma/(\gamma+1)}(\Omega)$ and $\mathbf{v}\cdot\mathbf{n} = 0$ satisfied on $\partial\Omega$ in the sense of traces, and if the following holds true:

1. Continuity equation is satisfied in the distributional sense[5]

$$\operatorname{div}\big(\varrho\mathbf{v}\big) = 0, \text{ in } \mathcal{D}'(\mathbb{R}^3).$$

2. For every $\boldsymbol{\varphi} \in C^\infty(\overline{\Omega}, \mathbb{R}^3)$, $\boldsymbol{\varphi}\cdot\mathbf{n} = 0$ at $\partial\Omega$

$$\int_\Omega \big(-\varrho(\mathbf{v}\otimes\mathbf{v}) : \nabla\boldsymbol{\varphi} + \mathbb{T}(\varrho, c, \nabla c, \nabla\mathbf{v}) : \nabla\boldsymbol{\varphi}\,\big)\,\mathrm{d}x + \sum_{k=1}^{2}\int_{\partial\Omega} \alpha(\mathbf{v}\cdot\boldsymbol{\tau}^{\boldsymbol{k}})(\boldsymbol{\varphi}\cdot\boldsymbol{\tau}^{\boldsymbol{k}})\mathrm{d}S = \int_\Omega \varrho\mathbf{g}\cdot\boldsymbol{\varphi}\;\mathrm{d}x.$$

[4]For slip boundary condition corresponding to the case $\alpha = 0$, we need to assume that the domain Ω is not axially symmetric.

[5]Note that we do not need to incorporate the notion of renormalized continuity equation in our definition, since density is regular enough.

3. For every $\varphi \in C^\infty(\overline{\Omega}, \mathbb{R})$

$$\int_\Omega \varrho \mathbf{v} \cdot \nabla c \, \varphi \, \mathrm{d}x = \int_\Omega \overset{+}{c} \varphi \, \mathrm{d}x$$

and

$$\int_\Omega \varrho \overset{+}{c} \varphi \, \mathrm{d}x = -\int_\Omega \varrho \frac{\partial f(\varrho, c)}{\partial c} \varphi + \nabla c \cdot \nabla \varphi \, \mathrm{d}x. \tag{2.9}$$

The main result of this chapter is the following.

Theorem 2.2. *Let $\gamma > 6$, $M_0 > 0$ and $\mathbf{g} \in L^\infty(\Omega, \mathbb{R}^3)$. Then there exists at least one weak solution to the system (2.1)–(2.4) such that $c \in [0,1]$ in Ω,*

$$\varrho \in L^\infty(\Omega), \quad \mathbf{v} \in W^{1,p}(\Omega, \mathbb{R}^3) \text{ and } c \in W^{2,p}(\Omega) \text{ for all } p < \infty. \tag{2.10}$$

As a corollary we obtain the following existence result, however losing the pointwise boundedness of the density.

Theorem 2.3. *If we assume only $\gamma > 3$ and $M_0 > 0$, $\mathbf{g} \in L^\infty(\Omega, \mathbb{R}^3)$. Then there exists at least one weak solution such that*

$$\varrho \in L^{3\gamma-6}(\Omega), \quad \mathbf{v} \in W^{1,2}(\Omega, \mathbb{R}^3), \quad \nabla c \in L^{\frac{6\gamma-12}{\gamma}}(\Omega). \tag{2.11}$$

Remark 2.4. Note that for $\gamma > 4$, according to Morrey's inequality the representative of the resulting concentration function c can be chosen to be continuous. In this case, the viscosity coefficients could depend on c in a suitable manner, but we omit it here.

The chapter is organized as follows. First we compute formal a priori estimate, it allows us to determine expected regularity of sought solutions. In Section 2.3, we deal with the approximative system, we construct regular approximative solutions together with required estimates in the dependence of approximative parameters. Finally we analyze the limit, showing the strong convergence of approximative densities. The approach is inspired by the corresponding works of Mucha and Pokorný [100, 101, 132]. The method is based on the fact that for the approximative densities ϱ_ϵ we are able to find such k that

$$\lim_{\epsilon \to 0} |\{\varrho_\epsilon > k\}| = 0. \tag{2.12}$$

The proof of Theorem 2.3 will be explained in Section 2.5. The results of this chapter are contained in article [7].

2.2 A priori bounds

Before the technical part of the proof, we will present here the a priori estimates on certain norms of the solution. All generic constants, which may depend on the given data as well, will be denoted by C, its values can vary from line to line or even in the same formula.

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^{2+\zeta}$ boundary. Assume that all the above mentioned hypotheses are satisfied with $\gamma > 3$ and that $\varrho, \mathbf{v}, \overset{+}{c}, c$ is a sufficiently smooth solution satisfying system (2.1)–(2.4) with boundary conditions (2.5)–(2.7). Then*

$$\|\mathbf{v}\|_{W^{1,2}(\Omega)} + \|\varrho\|_{L^{3\gamma-6}(\Omega)} + \|\nabla c\|_{L^{\frac{6\gamma-12}{\gamma}}(\Omega)} + \|\overset{+}{c}\|_2 + \|\varrho L'(c)\|_{L^{\frac{6\gamma-12}{3\gamma-4}}(\Omega)} \leq C, \tag{2.13}$$

where C may depend on the data, but is independent of the solution.

Remark 2.6. Note that $\frac{6\gamma-12}{\gamma} > 2$, for $\gamma > 3$ and that from the bound of the last norm on the left-hand side, we immediately deduce that $c \in [0,1]$ a.e. on the set $\{\varrho > 0\}$. Moreover, if $\gamma > 4$, then according to the Sobolev embedding c is continuous function and we conclude from the maximum principle for harmonic functions that in fact $c \in [0,1]$ a.e. in Ω.

Proof. First, multiplying the momentum equation by the velocity field $\mathbf{v}$ yields with use of Korn's inequality from Theorem 4.4 and boundary condition

$$-\int_\Omega p \operatorname{div} \mathbf{v} \,\mathrm{d}x + C\int_\Omega |\nabla \mathbf{v}|^2 \,\mathrm{d}x \leq \int_\Omega -\varrho\overset{+}{c}\nabla c \cdot \mathbf{v} - \varrho\frac{\partial f}{\partial c}\nabla c \cdot \mathbf{v} + \varrho \mathbf{g}\cdot\mathbf{v} \,\mathrm{d}x, \tag{2.14}$$

where we have used equation (2.4) as well. Further, we conclude from the continuity equation that

$$\int_\Omega p \operatorname{div} \mathbf{v} + \operatorname{div}(\varrho f \mathbf{v}) \,\mathrm{d}x = \int_\Omega \varrho \frac{\partial f}{\partial c}\nabla c \cdot \mathbf{v} \,\mathrm{d}x, \tag{2.15}$$

and according to the constitutive equation for $\overset{+}{c}$ we get

$$\int_\Omega \big(\overset{+}{c}\big)^2 \,\mathrm{d}x = \int_\Omega \varrho\overset{+}{c}\nabla c \cdot \mathbf{v} \,\mathrm{d}x. \tag{2.16}$$

Thus, summing up (2.14)–(2.16) yields for $p > \frac{6}{5}$

$$\int_\Omega |\nabla \mathbf{v}|^2 + \big(\overset{+}{c}\big)^2 \,\mathrm{d}x \leq C\int_\Omega \varrho \mathbf{g}\cdot\mathbf{v} \,\mathrm{d}x \leq C\Big(1 + \|\varrho\|^2_{L^{\frac{6}{5}}(\Omega)}\Big) \leq C\Big(1 + \|\varrho\|^{\frac{p}{3(p-1)}}_{L^p(\Omega)}\Big).$$

In order to bound the density by means of the Bogovskii estimates, we need ∇c in $L^q(\Omega)$, $q > 2$. This can be deduced from the constitutive relation for $\overset{+}{c}$. We state this for purpose of future references more generally in the following lemma.

Lemma 2.7. *Suppose that $f(\varrho, c)$ is as above, and $\overset{+}{c} \in L^q(\Omega)$ and $\varrho \in L^p(\Omega)$ with $q \geq 2$, $p > 3$, $q < \frac{3p}{p-3}$ satisfy the equation (2.4) with boundary condition (2.7). Then*

$$\Big\|\varrho\frac{\partial f}{\partial c}\Big\|_{L^{\frac{pq}{p+q}}(\Omega)} + \|\varrho\overset{+}{c}\|_{\frac{pq}{p+q}} + \|\Delta c\|_{L^{\frac{pq}{p+q}}(\Omega)} + \|\nabla c\|_{L^{\frac{3pq}{3p+3q-pq}}(\Omega)}$$
$$\leq C\Big(\|\varrho\|_{L^p(\Omega)}\big(\|\overset{+}{c}\|_q + 1\big) + 1\Big).$$

Proof of Lemma 2.7. Let us test the corresponding equation by $F\big(L'(c)\big)$, for increasing function F with growth $\zeta F(\zeta) \sim |\zeta|^{\beta+1}$ with some $\beta > 0$, so (recall that L is convex)

$$
\begin{aligned}
\left\|\varrho^{1/(\beta+1)} L'(c)\right\|_{L^{\beta+1}(\Omega)}^{\beta+1} &\leq \int_\Omega |\nabla c|^2 F'\big(L'(c)\big) L''(c) + \varrho L'(c) F\big(L'(c)\big)\,\mathrm{d}x \\
&= \int_\Omega \varrho\big(-\overset{+}{c} - a\log\varrho - b'(c)\big) F\big(L'(c)\big)\,\mathrm{d}x \\
&\leq C \int_\Omega \varrho^{\frac{\beta}{\beta+1}} |L'(c)|^\beta \left(\big|\overset{+}{c}\big| \varrho^{\frac{1}{\beta+1}} + \varrho^{\frac{1}{\beta+1}} |\log\varrho|\right) + \varrho\, |L'(c)|^\beta\, |c|\,\mathrm{d}x \\
&\leq C\Bigg(\left\|\varrho^{\frac{1}{\beta+1}} L'(c)\right\|_{L^{\beta+1}}^{\beta} \left(\big\|\overset{+}{c}\big\|_{L^q} \|\varrho\|_{L^{\frac{q}{q-(\beta+1)}}}^{\frac{1}{\beta+1}} + \left(\int_\Omega \varrho\, |\log\varrho|^{\beta+1}\,\mathrm{d}x\right)^{\frac{1}{\beta+1}} \right) \\
&\qquad\qquad + \left\|\varrho^{\frac{1}{\beta+1}} L'(c)\right\|_{L^{\beta+1}}^{\beta+\frac12} \Bigg)
\end{aligned}
$$

which yields

$$
\left\|\varrho^{1/(\beta+1)} L'(c)\right\|_{L^{\beta+1}(\Omega)} \leq\ C\left(1 + \big\|\overset{+}{c}\big\|_{L^q} \|\varrho\|_{L^{\frac{q}{q-(\beta+1)}}}^{\frac{1}{\beta+1}} + \left(\int_\Omega \varrho\, |\log\varrho|^{\beta+1}\,\mathrm{d}x\right)^{\frac{1}{\beta+1}} \right).
$$

Now we fix the exponent β such that $\frac{q}{q-(\beta+1)} = p \in \big(3,\infty\big)$, hence

$$
\left\|\varrho^{p/(pq-q)} L'(c)\right\|_{L^{\frac{q(p-1)}{p}}} \leq C\left(1 + \big\|\overset{+}{c}\big\|_{L^q} \|\varrho\|_{L^p}^{\frac{p}{q(p-1)}} + \left(\int_\Omega \varrho\, |\log\varrho|^{\frac{(p-1)q}{p}}\,\mathrm{d}x\right)^{\frac{p}{q(p-1)}} \right).
$$

Thus,

$$
\begin{aligned}
\|\varrho L'(c)\|_{L^{\frac{pq}{p+q}}(\Omega)} &= \left\|\varrho^{p/(pq-q)} \varrho^{(pq-q-p)/(pq-q)} L'(c)\right\|_{L^{\frac{pq}{q+p}}} \\
&\leq \left\|\varrho^{p/(pq-q)} L'(c)\right\|_{L^{\frac{q(p-1)}{p}}} \left\|\varrho^{(pq-q-p)/(pq-q)}\right\|_{L^{\frac{pq(q-1)}{pq-p-q}}} \\
&\leq C\, \|\varrho\|_{L^p}^{\frac{pq-p-q}{q(p-1)}} \left(1 + \big\|\overset{+}{c}\big\|_{L^q} \|\varrho\|_{L^p}^{\frac{p}{q(p-1)}} + \left(\int_\Omega \varrho\, |\log\varrho|^{\frac{(p-1)q}{p}}\,\mathrm{d}x\right)^{\frac{p}{q(p-1)}} \right) \\
&\leq C\, \|\varrho\|_{L^p}^{\frac{pq-p-q}{q(p-1)}} \left(1 + \big\|\overset{+}{c}\big\|_{L^q} \|\varrho\|_{L^p}^{\frac{p}{q(p-1)}} + \|\varrho\|_{L^p}^{\frac{p}{q(p-1)}} \right) \\
&\leq C\left(\left(\big\|\overset{+}{c}\big\|_q + 1\right) \|\varrho\|_{L^p(\Omega)} + 1 \right).
\end{aligned}
$$

For the other terms we have

$$
\begin{aligned}
\|\varrho b'(c)\|_{L^{\frac{pq}{p+q}}(\Omega)} + \|\varrho\log\varrho\|_{L^{\frac{pq}{p+q}}(\Omega)} &\leq C(1 + \|\varrho\|_{L^p(\Omega)}), \\
\left\|\varrho\overset{+}{c}\right\|_{\frac{pq}{p+q}} &\leq \|\varrho\|_{L^p(\Omega)} \big\|\overset{+}{c}\big\|_q.
\end{aligned}
$$

Therefore, using the classical elliptic estimates on equation $\Delta c = \varrho\overset{+}{c} + \varrho\frac{\partial f}{\partial c}$, together with embedding $W^{1,\frac{pq}{p+q}}(\Omega) \hookrightarrow L^{\frac{3pq}{3p+3q-pq}}(\Omega)$ we get, see Theorem 4.11 and the remark below,

$$
\begin{aligned}
&\|\nabla c\|_{L^{\frac{3pq}{3p+3q-pq}}(\Omega)} \\
&\qquad \leq C\left(\left\| \varrho\frac{\partial f}{\partial c} + \varrho\overset{+}{c} \right\|_{L^{\frac{qp}{q+p}}(\Omega)} + \|c\|_{L^s(\tilde\Omega)} \right) \leq C\left(\left(\big\|\overset{+}{c}\big\|_q + 1\right) \|\varrho\|_{L^p(\Omega)} + 1 \right),
\end{aligned}
$$

where for $\widetilde{\Omega}$ we can take $\{\varrho > \varrho_0\}$ which has positive measure and on which $c \in [0,1]$ a.e. according to the logarithmic terms in $L'(c)$. This completes the proof of Lemma 2.7. $\square$

Applying Lemma 2.7 with $q = 2$ yields $\|\nabla c\|_{L^{\frac{6p}{6+p}}(\Omega)} \leq C\big(1 + \|\varrho\|_{L^p(\Omega)}^{(7p-6)/(6p-6)}\big)$. Now, we are ready to perform the Bogovskii estimates, id est to test the momentum equation by[6]

$$\mathbf{\Phi} = \mathcal{B}\left[\varrho^\alpha - \{\varrho^\alpha\}_\Omega\right],$$

where $\alpha > 0$ will be specified later, and $\mathcal{B}$ is the Bogovskii operator. Theorem 4.13 implies that $\|\nabla\mathbf{\Phi}\|_{L^p} \leq C\|\varrho^\alpha\|_{L^p}$, $\mathbf{\Phi}|_{\partial\Omega} = \mathbf{0}$; we obtain

$$\int_\Omega p(\varrho, c)\varrho^\alpha \,\mathrm{d}x \leq \int_\Omega \big(-(\varrho\mathbf{v}\otimes\mathbf{v}) : \nabla\mathbf{\Phi} + \mathbb{S}(c, \nabla\mathbf{v}) : \nabla\mathbf{\Phi} - \varrho\mathbf{g}\cdot\mathbf{\Phi}\big)\,\mathrm{d}x$$
$$+ C\int_\Omega |\nabla c|^2\,|\nabla\mathbf{\Phi}|\;\mathrm{d}x + \int_\Omega p(\varrho,c)\,\{\varrho^\alpha\}_\Omega\,\mathrm{d}x.$$

The terms on the left-hand side of the inequality have sign and provide the desired estimate of $\varrho^{\gamma+\alpha}$, if the right-hand side will be estimated, thus we set $p = \gamma + \alpha$. We will present only the most difficult and restrictive terms.

$$\begin{aligned}\int_\Omega |\nabla c|^2\,|\nabla\mathbf{\Phi}|\,\mathrm{d}x &\leq \|\nabla c\|^2_{L^{\frac{6p}{6+p}}(\Omega)}\,\|\nabla\mathbf{\Phi}\|_{L^{\frac{3p}{2p-6}}(\Omega)} \\ &\leq C\Big(1 + \|\varrho\|_{L^p(\Omega)}^{\frac{7p-6}{3p-3}}\,\|\varrho^\alpha\|_{L^{\frac{3p}{2p-6}}(\Omega)}\Big) \leq C\Big(1 + \|\varrho\|_{L^p(\Omega)}^{\frac{7p-6}{3p-3}}\,\|\varrho\|_{L^p(\Omega)}^{\frac{3\alpha p-2p+6}{3(p-1)}}\Big) \\ &\leq C\Big(1 + \|\varrho\|_{L^p(\Omega)}^{(5p+3\alpha p)/(3p-3)}\Big),\end{aligned} \tag{2.17}$$

provided $\frac{3p\alpha}{2p-6} \leq p$, or equivalently

$$0 < \alpha \leq 2(\gamma - 3), \tag{2.18}$$

which yields the restriction $\gamma > 3$. The condition $\frac{(\gamma+\alpha)(5+3\alpha)}{3(\gamma+\alpha)-3} < \gamma + \alpha$ is satisfied even for all $\gamma > \frac{8}{3}$, so we can put this term to the left-hand side. Further, provided that $\alpha < 2\gamma - 3$ we have

$$\begin{aligned}\int_\Omega |(\varrho\mathbf{v}\otimes\mathbf{v}) : \nabla\mathbf{\Phi}|\,\mathrm{d}x &\leq \|\mathbf{v}\|^2_{L^6(\Omega)}\,\|\varrho\nabla\mathbf{\Phi}\|_{L^{3/2}(\Omega)} \\ &\leq C\big(1 + \|\varrho\|_{L^p(\Omega)}^{\frac{p}{3(p-1)}}\big)\,\|\varrho\|_{L^{\frac{3p}{2p-3\alpha}}(\Omega)}\,\|\nabla\mathbf{\Phi}\|_{L^{p/\alpha}(\Omega)} \\ &\leq C\Big(1 + \|\varrho\|_{L^p(\Omega)}^{\frac{p}{3(p-1)}+\alpha}\Big)\,\|\varrho\|_{L^p(\Omega)}^{\frac{p+3\alpha}{3(p-1)}},\end{aligned} \tag{2.19}$$

where the condition $\frac{2(\gamma+\alpha)+3\alpha}{3(\gamma+\alpha-1)} + \alpha < \gamma + \alpha$ is less restrictive since it requires only $5 + 3\alpha < 3(\gamma+\alpha) \Rightarrow \gamma > \frac{5}{3}$. The other terms can be estimated similarly so we get taking maximal possible value of $\alpha = 2(\gamma - 3)$ that

$$\|\varrho\|_{L^{3\gamma-6}(\Omega)} \leq C.$$

Using this in the already derived estimates for $\mathbf{v}, c$ and $\overset{+}{c}$ yields the result of Lemma 2.5. $\square$

[6] Recall the notation $\{g\}_\Omega = \frac{1}{|\Omega|}\int_\Omega g\;\mathrm{d}x$.

Now, we will show that for $\gamma > 6$ we can expect principally better regularity of the solutions, this coheres with the fact that in this case we can take according to (2.18) $\alpha > \gamma$, so $p(\varrho, c) \in L^s(\Omega)$, for some $s > 2$.

Lemma 2.8. *For $\gamma > 6$ we have for solutions to (2.1)–(2.7) for any $1 < p < +\infty$*

$$\|\mathbf{v}\|_{W^{1,p}(\Omega)} + \|\varrho\|_{L^\infty(\Omega)} + \|\nabla c\|_{L^\infty(\Omega)} + \left\|\overset{+}{c}\right\|_{L^\infty(\Omega)} + \|\varrho L'(c)\|_{L^\infty(\Omega)} \le C_p,$$

and $c \in [0, 1]$ a.e. in Ω.

Proof. First, since $\gamma > 6$ we observe certain smoothing effect of (2.3) and (2.4). In what follows, we will repeatedly use Hölder's inequality in the third equation and Lemma 2.7. Indeed, since $\varrho\mathbf{v}\cdot\nabla c = \overset{+}{c}$, and $\varrho \in L^{3\gamma-6}$, $\mathbf{v} \in L^6$ and $\nabla c \in L^{\frac{6\gamma-12}{\gamma}}$ we get $\overset{+}{c} \in L^{\frac{3\gamma-6}{\gamma}}$, $(\frac{1}{3\gamma-6} + \frac{1}{6} + \frac{\gamma}{6\gamma-12} < \frac{1}{2})$, and applying Lemma 2.7 $\nabla c \in L^{\gamma-2}$, which can be again plugged into the third equation in order to get $\overset{+}{c} \in L^{\frac{6\gamma-12}{\gamma+6}}$, and again $\nabla c \in L^{\frac{6\gamma-12}{12-\gamma}}$, at least for $6 < \gamma < 12$, etc. This procedure can be repeated until $\nabla c \in L^\infty$, since there exists no reasonable finite solution to the following system of algebraic equations, where P corresponds to the expected integrability of ∇c, Q to $\overset{+}{c}$ and $\frac{3Q(\gamma-2)}{3\gamma+Q-6}$ corresponds to $\varrho\overset{+}{c}$

$$\frac{1}{3\gamma-6} + \frac{1}{6} + \frac{1}{P} = \frac{1}{Q}, \qquad P = \frac{3Q(\gamma-2)}{3\gamma-6-Q\gamma-Q}. \tag{2.20}$$

So we get that $P \to +\infty$, and $Q \to \frac{6\gamma-12}{\gamma} > 3$, id est for any $\gamma > 6$ after finite number of such steps we have $\nabla c \in L^\infty(\Omega)$, $\overset{+}{c} \in L^{\frac{6\gamma-12}{\gamma}}(\Omega)$, and $\nabla^2 c \in L^{\frac{6\gamma-12}{\gamma+2}}(\Omega)$. Thus, to summarize

$$\begin{aligned}\|\mathbf{v}\|_{W^{1,2}(\Omega)} &+ \|\varrho\|_{L^{3\gamma-6}(\Omega)} + \|\nabla c\|_{L^\infty(\Omega)} \\ &+ \left\|\overset{+}{c}\right\|_{L^{\frac{6\gamma-12}{\gamma}}(\Omega)} + \left\|\nabla^2 c\right\|_{L^{\frac{6\gamma-12}{\gamma+2}}(\Omega)} + \|\varrho L'(c)\|_{L^{\frac{6\gamma-12}{\gamma+2}}(\Omega)} \le C.\end{aligned}$$

From the last norm we can deduce $c \in [0, 1]$ a.e. on $\{\varrho > 0\}$. On the other hand, c is continuous in Ω and on the set $\{\varrho = 0\}$ it satisfies the Laplace equation, and therefore maximum principle. Thus,[7] $c \in [0, 1]$ a.e. in Ω.

Now, we will decompose the velocity field, using the Helmholtz decomposition from Theorem 4.19 $\mathbf{v} = P_H\mathbf{v} + \nabla P_\nabla\mathbf{v}$, so consequently $P_H\mathbf{v}$ satisfies the following overdetermined system

$$\begin{aligned}\operatorname{curl} P_H\mathbf{v} = \operatorname{curl}\mathbf{v} &= \boldsymbol{\omega}, \\ \operatorname{div} P_H\mathbf{v} &= 0 \text{ in } \Omega, \\ P_H\mathbf{v}\cdot\mathbf{n} &= 0 \text{ on } \partial\Omega,\end{aligned}$$

where we have denoted the vorticity of the velocity field by $\boldsymbol{\omega} = \operatorname{curl}\mathbf{v}$. According to Theorem 4.14 we have

$$\|\nabla P_H\mathbf{v}\|_{L^p(\Omega)} \le C\,\|\boldsymbol{\omega}\|_{L^p(\Omega)},$$

[7] More precisely, since c is continuous, the set $U = \{c \notin [0, 1]\}$ is open. Considering any ball $B(r, x_0) \subset U$ we get that $\varrho = 0$ a.e. in $B(r, x_0)$, hence c satisfies $\Delta c = 0$ in $B(r, x_0)$ and can not reach neither maximum, nor minimum within this ball.

hence in order to bound $\mathbf{v}$ we can concentrate on $\boldsymbol{\omega}$ and $P_{\nabla}\mathbf{v}$.

Using the fact that we work with slip boundary conditions[8] we can deduce from the momentum equation the following Stokes-like problem for vorticity $\boldsymbol{\omega}$, for the derivation of the boundary conditions see [98]

$$-\mu\Delta\boldsymbol{\omega} = -\operatorname{curl}(\varrho\mathbf{v}\cdot\nabla\mathbf{v}) - \operatorname{curl}\big(\Delta c\nabla c\big) + \operatorname{curl}(\varrho\mathbf{g}) \text{ in } \Omega, \tag{2.21}$$

$$\boldsymbol{\omega}\cdot\boldsymbol{\tau}^1 = -(2\chi_2 - \frac{\alpha}{\mu})\mathbf{v}\cdot\boldsymbol{\tau}^2 \text{ on } \partial\Omega, \tag{2.22}$$

$$\boldsymbol{\omega}\cdot\boldsymbol{\tau}^2 = (2\chi_1 - \frac{\alpha}{\mu})\mathbf{v}\cdot\boldsymbol{\tau}^1 \text{ on } \partial\Omega, \tag{2.23}$$

$$\operatorname{div}\boldsymbol{\omega} = 0 \text{ on } \partial\Omega, \tag{2.24}$$

where χ_k are the curvatures related to the vectors $\boldsymbol{\tau}^k$. Note, that $\nabla c\cdot\mathbf{n}|_{\partial\Omega} = 0$ yields $\mathbf{n}\cdot(\nabla c\otimes\nabla c)\cdot\boldsymbol{\tau}^k = 0$ on $\partial\Omega$. Now, we will show that $\|\nabla\boldsymbol{\omega}\|_{L^p(\Omega)} \leq C$, for some $p > 1$.

First, we have $\mathbf{v}\cdot\boldsymbol{\tau}^k\big|_{\partial\Omega} \in W^{\frac{1}{2},2}(\partial\Omega)$, so we control $\boldsymbol{\omega}$ on the boundary. Further, $\nabla\mathbf{v}\nabla c \in L^2(\Omega)$ $\varrho\mathbf{g} \in L^{3\gamma-6}(\Omega)$, $\varrho\mathbf{v}\cdot\nabla\mathbf{v} \in L^p(\Omega)$, for any $p \in \big(1, \frac{6\gamma-12}{4\gamma-6}\big)$, and finally

$$\|\Delta c\nabla c\|_{L^{\frac{6\gamma-12}{\gamma+2}}(\Omega)} \leq \|\Delta c\|_{L^{\frac{6\gamma-12}{\gamma+2}}(\Omega)} \|\nabla c\|_{L^\infty(\Omega)} \leq C.$$

According to the fact that $1 < \frac{3\gamma-6}{2\gamma-3} < \frac{6\gamma-12}{\gamma+2}$ even for all $\gamma > \frac{8}{3}$, we get

$$\boldsymbol{\omega} \in W^{1,\frac{3\gamma-6}{2\gamma-3}}(\Omega).$$

Now, we will proceed in the same manner as in [100], we define the effective viscous flux

$$G = p(\varrho, c) - (2\mu+\lambda)\Delta P_{\nabla}\mathbf{v}, \tag{2.25}$$

observe

$$\nabla G = -\varrho\mathbf{v}\cdot\nabla\mathbf{v} + \mu\Delta P_H\mathbf{v} + (\varrho\mathbf{g} + \Delta c\nabla c)$$

and show that $G \in W^{1,\frac{3\gamma-6}{2\gamma-3}}(\Omega) \hookrightarrow L^{\frac{3\gamma-6}{\gamma-1}}(\Omega)$. This further yields

$$\varrho^\gamma \in L^{12/5}(\Omega), \quad \mathbf{v} \in W^{1,12/5}(\Omega) \hookrightarrow L^{12}(\Omega),$$

and after more iterations $\varrho \in L^\infty(\Omega)$, $\mathbf{v} \in W^{1,p}(\Omega)$, $c \in W^{1,p}(\Omega)$, for arbitrary $p \in (1,\infty)$ as well. □

[8] Indeed, from 2.6 we obtain with use of 2.5 e.g.

$$-\partial_{\mathbf{n}}(\mathbf{v}_{\boldsymbol{\tau}^2}) = \frac{\alpha}{\mu}\mathbf{v}\cdot\boldsymbol{\tau}^2 - \sum_{i,j} v_j\big(n_i\partial_i(\tau_j^2) + \tau_i^2\partial_i(n_j)\big),$$

which leads directly to (2.22), and analogously for (2.23).

2.3 Approximation

2.3.1 Approximation scheme

In this section we define a problem approximating the original one and prove the existence of the corresponding solutions. We introduce $m = M_0/|\Omega|$, $\epsilon > 0$, a smooth cut-off function $K(\varrho)$

$$K(\varrho) = \begin{cases} 1, & \text{for } \varrho \le k-1 \\ \in (0,1), & \text{for } \varrho \in (k-1,k) \\ 0, & \text{for } \varrho \ge k, \end{cases}$$

a "regularized logarithm" which is a function $l_\epsilon \in C^1([0,\infty))$ which is bounded from below by $\log(\sqrt[t]{\epsilon}) - 1$ ($t > 1$ will be specified later) and

$$l_\epsilon(s) = \begin{cases} \log(s), & \text{for } s \ge \sqrt[t]{\epsilon}, \\ \text{convex, non-decreasing} & \text{for } s < \sqrt[t]{\epsilon}, \end{cases}$$

with[9]

$$\begin{aligned} &0 \le \sqrt[t]{\epsilon}\big(2l'_\epsilon(s) + s l''_\epsilon(s)\big) \le C, \\ &0 \le s\big(2l'_\epsilon(s) + s l''_\epsilon(s)\big) \le C \text{ for a.a. } s \in [0,\infty), \end{aligned} \tag{2.26}$$

where C is independent of ϵ; further we denote the approximated free energy

$$f_\epsilon(\varrho, c) = \Gamma(\varrho) + (ac+d)l_\epsilon(\varrho) + L_\epsilon(c) + b(c),$$

where we define $L_\epsilon(c) = \int\limits_0^c l_\epsilon(s) - l_\epsilon(1-s)\mathrm{d}s$ for $c \in [0,1]$, and then extend it to whole $\mathbb{R}$ as a convex function with $\|L'_\epsilon\|_{L^\infty(\Omega)} \le -C\log\sqrt[t]{\epsilon}$, $\Gamma(\varrho) = \frac{\varrho^{\gamma-1}}{\gamma-1}$ and approximated pressure

$$p_\epsilon(\varrho, c) = P_b(\varrho) + (ac+d)\int_0^\varrho K(s)\frac{\mathrm{d}}{\mathrm{d}s}\big(s^2 l'_\epsilon(s)\big)\,\mathrm{d}s,$$

where $P_b(\varrho) = \int_0^\varrho \gamma s^{\gamma-1}K(s)\,\mathrm{d}s$.

Our approximation problem then reads

$$\epsilon\varrho + \operatorname{div}(K(\varrho)\varrho\mathbf{v}) = \epsilon\Delta\varrho + \epsilon K(\varrho)m, \tag{2.27}$$

$$\begin{aligned} \frac{1}{2}\operatorname{div}(K(\varrho)\varrho\mathbf{v}\otimes\mathbf{v}) + \frac{1}{2}K(\varrho)\varrho\mathbf{v}\cdot\nabla\mathbf{v} - \mu\Delta\mathbf{v} - (\mu+\lambda)\nabla(\operatorname{div}\mathbf{v}) + \nabla p_\epsilon(\varrho, c) \\ = K(\varrho)\varrho\mathbf{g} + \operatorname{div}\Big(\nabla c\otimes\nabla c - \frac{|\nabla c|^2}{2}\mathbb{I}\Big) - a\nabla c\int_0^\varrho sK'(s)\mathrm{d}s, \end{aligned} \tag{2.28}$$

$$K(\varrho)\varrho\mathbf{v}\cdot\nabla c = \overset{+}{c}, \tag{2.29}$$

$$K(\varrho)\varrho\overset{+}{c} = \Delta c - K(\varrho)\varrho\frac{\partial f_\epsilon}{\partial c} + \epsilon\varrho K(\varrho)L'(c). \tag{2.30}$$

Moreover, we supply the first equation with additional boundary condition

$$\nabla\varrho\cdot\mathbf{n} = 0. \tag{2.31}$$

[9] We can get such a function e.g. by replacing the logarithm by a suitable affine function for small arguments.

2.3.2 Existence of approximate solutions for fixed parameters

Proposition 2.9. *Let $\epsilon > 0$. Suppose that the assumptions of Theorem 2.2 are satisfied, then there exists at least one solution $\varrho_\epsilon, \mathbf{v}_\epsilon, \overset{+}{c}_\epsilon, c_\epsilon$ to the system (2.27)–(2.30) with (2.5)–(2.8), (2.31). Moreover, we have with $1 < q < +\infty$ the following estimates independent of ϵ*

$$\|\varrho_\epsilon\|_{L^\infty(\Omega)} + \|\mathbf{v}_\epsilon\|_{W^{1,q}(\Omega)} + \epsilon\,\|\nabla\varrho_\epsilon\|^2_{L^2(\Omega)} + \epsilon\,\big\|(K(\varrho_\epsilon)\varrho_\epsilon)^{1/2}L'(c_\epsilon)\big\|_{L^2(\Omega)} \le C(k),$$

$$\|p_\epsilon\|_{L^2} + \big\|\overset{+}{c}_\epsilon\big\|_{L^2} + \|\nabla c_\epsilon\|_{L^{\frac{6\gamma}{3+\gamma}}} + \|\mathbf{v}_\epsilon\|_{W^{1,2}} + \Big\|\big(K(\varrho_\epsilon)\varrho_\epsilon\big)^{\frac{\gamma+1}{2\gamma}} L'_\epsilon(c_\epsilon)\Big\|_{L^{\frac{2\gamma}{\gamma+1}}} \le C. \tag{2.32}$$

Proof. The existence of solutions for the approximative system will be deduced by means of Theorem 4.9.

We define space

$$\mathbf{M}_q = \left\{\mathbf{w} \in \left(W^{1,q}(\Omega), \mathbb{R}^3\right), \mathbf{w}\cdot\mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\right\},$$

and find the solution as a fixed point to certain mapping on $\mathbf{M}_p$ with $3 < p \le \infty$, see [37, 118] for similar considerations for the Navier–Stokes system. Let us first concentrate on the continuity equation.

Lemma 2.10. *The solution operator $\mathcal{S}_1(\mathbf{v}) = \varrho$ of problem*

$$\epsilon\varrho + \operatorname{div}(K(\varrho)\varrho\mathbf{v}) = \epsilon\Delta\varrho + \epsilon K(\varrho)m \text{ in } \Omega,$$
$$\nabla\varrho\cdot\mathbf{n} = 0 \text{ on } \partial\Omega$$

is for $p > 3$ a well-defined continuous operator from $\mathbf{M}_p$ to $W^{2,p}(\Omega)$. Moreover, we have $\varrho \ge 0$, $\int_\Omega \varrho \,\mathrm{d}x \le M_0$, and

$$\|\varrho\|_{W^{2,p}} \le C(k,\epsilon)(1 + \|\mathbf{v}\|_{W^{1,p}(\Omega)}).$$

Proof. The proof is a quite standard analysis, see e.g. [101, Lemma 2], [118, Proposition 4.29], for the continuity of the operator see [80, Lemma 2.3], and for the regularity see Theorem 4.11. We recall here only the idea how to obtain the estimates. First, considering the subset $\{\varrho < 0\} \subset \Omega$ we get $\varrho \ge 0$ a.e. in Ω, then integrating the approximate continuity equation over Ω yields

$$\int_\Omega \varrho \,\mathrm{d}x = m\int_\Omega K(\varrho)\,\mathrm{d}x,$$

so $K(\varrho) = 1$ a.e. in Ω. Further, testing the continuity equation by ϱ yields, see Theorem 4.11

$$\begin{aligned}\epsilon\int_\Omega \varrho^2 + |\nabla\varrho|^2\,\mathrm{d}x - \epsilon\int_\Omega K(\varrho)\varrho m\,\mathrm{d}x \\ \le -\int_\Omega \varrho\operatorname{div}\big(K(\varrho)\varrho\mathbf{v}\big)\,\mathrm{d}x = \int_\Omega \mathbf{v}\cdot\nabla\varrho K(\varrho)\varrho\,\mathrm{d}x \\ = \int_\Omega \mathbf{v}\cdot\nabla\Big(\int_0^\varrho K(s)s\mathrm{d}s\Big)\,\mathrm{d}x \le C\,\big\|K(\varrho)\varrho^2\big\|_{L^2(\Omega)}\|\nabla\mathbf{v}\|_{L^2(\Omega)},\end{aligned}$$

since the last term on the left-hand side can be easily bounded we get

$$\epsilon\,\|\nabla\varrho\|^2_{L^2(\Omega)} \le C\big(1 + \big\|K(\varrho)\varrho^2\big\|_{L^2(\Omega)}\|\nabla\mathbf{v}\|_{L^2(\Omega)}\big). \tag{2.33}$$

□

Similarly, for the last two equations we have

Lemma 2.11. *The solution operator $\mathcal{S}_2(\mathbf{v}) = c$ of problem*

$$\begin{aligned}
\overset{+}{c} &= K(\varrho)\varrho\nabla c\cdot\mathbf{v},\\
\Delta c - K(\varrho)\varrho\epsilon L'(c) &= K(\varrho)\varrho\overset{+}{c} + K(\varrho)\varrho\frac{\partial f_\epsilon}{\partial c} \text{ in } \Omega, \text{ where } \varrho = \mathcal{S}_1(\mathbf{v}),\\
\nabla c\cdot\mathbf{n} &= 0 \text{ on } \partial\Omega
\end{aligned}$$

is for $p > 3$ a well-defined continuous operator from $\mathbf{M}_p$ to $W^{3,p}(\Omega)$. Moreover, $c \in [0,1]$, and

$$\|c\|_{W^{2,\frac{2q}{2+q}}} + \|\nabla c\|_{L^{\frac{6q}{6+q}}} \leq C\big(\|\overset{+}{c}\|_{L^2}\,\|K(\varrho)\varrho\|_{L^q} + 1\big).$$

Proof. The proof is quite similar to the one of Lemma 2.10, for the estimates we proceed analogously as in the proof of Lemma 2.7. For constructing the solution we use again the Leray–Schauder fixed point theorem. We consider for fixed density $\varrho \in W^{2,p}(\Omega)$ a mapping defined on $W^{2,p}(\Omega)$, $c \mapsto z$ as a solution operator to the problem

$$\begin{aligned}
\overset{+}{c} &= K(\varrho)\varrho\nabla c\cdot\mathbf{v},\\
\Delta z - K(\varrho)\varrho\epsilon L'(z) &= K(\varrho)\varrho\overset{+}{c} + K(\varrho)\varrho\frac{\partial f_\epsilon}{\partial c}(\varrho, c) \text{ in } \Omega,\\
\nabla z\cdot\mathbf{n} &= 0 \text{ on } \partial\Omega.
\end{aligned} \tag{2.34}$$

The second equation is for $\epsilon > 0$ strictly elliptic, furthermore, its right-hand side belongs to $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$, in particular we deduce from Theorem 4.11 that $K(\varrho)\varrho L'(c) \in L^p(\Omega)$ and $\nabla z \in W^{1,p}(\Omega)$, for any $p < \infty$. Therefore, since ϱ and z are continuous we find $K(\varrho)\varrho L''(z) \in L^p(\Omega)$, put the corresponding term to the right-hand side, observe that in fact $K(\varrho)\varrho\overset{+}{c} + K(\varrho)\varrho\frac{\partial f_\epsilon}{\partial c} + K(\varrho)\varrho\epsilon L'(z) \in W^{1,p}(\Omega)$, and so $z \in W^{3,p}(\Omega)$. The corresponding mapping is compact by the same reasons as above. In order to get the desired estimate independent of ϵ, we test the equation

$$\Delta c - K(\varrho)\varrho\big(L'_\epsilon(c) + \epsilon L'(c)\big) = K(\varrho)\varrho\overset{+}{c} + K(\varrho)\varrho\big(b'(c) + al_\epsilon(\varrho)\big)$$

by $F(L'_\epsilon(c) + \epsilon L'(c))$, where F is increasing function such that $\zeta F(\zeta) \sim |\zeta|^{\beta+1}$, $|F| \sim |\zeta|^\beta$, for some $\beta > 0$. Note that $L'_\epsilon(c) + \epsilon L'(c)$ is non-decreasing function, so we get

$$\begin{aligned}
\epsilon\,\big\|(K(\varrho)\varrho)^{1/(\beta+1)}L'(c)\big\|^{\beta+1}_{L^{\beta+1}(\Omega)} &+ \big\|(K(\varrho)\varrho)^{1/(\beta+1)}L'_\epsilon(c)\big\|^{\beta+1}_{L^{\beta+1}(\Omega)}\\
\leq \bigg|\int_\Omega K(\varrho)\varrho\overset{+}{c}F\big(L'_\epsilon(c) + \epsilon L'(c)\big)\,\mathrm{d}x\bigg|&\\
+ \bigg|\int_\Omega K(\varrho)\varrho\big(b'(c) + al_\epsilon(\varrho)\big)F\big(L'_\epsilon(c) + \epsilon L'(c)\big)\,\mathrm{d}x\bigg|&,
\end{aligned} \tag{2.35}$$

hence

$$\begin{aligned}
\epsilon\,\Big\|(K(\varrho)\varrho)^{\frac{1}{\beta+1}}L'(c)\Big\|_{L^{\beta+1}(\Omega)} &+ \Big\|(K(\varrho)\varrho)^{\frac{1}{\beta+1}}L'_\epsilon(c)\Big\|_{L^{\beta+1}(\Omega)}\\
&\leq C\Big(\|\overset{+}{c}\|_p\,\|K(\varrho)\varrho\|_{L^{\frac{p}{p-(\beta+1)}}(\Omega)} + 1\Big)
\end{aligned} \tag{2.36}$$

and using the classical elliptic estimates we obtain as in Lemma 2.7

$$\|c\|_{W^{2,\frac{pq}{p+q}}} + \|\nabla c\|_{L^{\frac{3pq}{3p+3q-pq}}} \leq C\big(\|\overset{+}{c}\|_{L^p} \|K(\varrho)\varrho\|_{L^q} + 1\big),$$

especially for $p = 2$,

$$\|c\|_{W^{2,\frac{2q}{2+q}}} + \|\nabla c\|_{L^{\frac{6q}{6+q}}} \leq C\big(\|\overset{+}{c}\|_{L^2} \|K(\varrho)\varrho\|_{L^q} + 1\big).$$

The issue of existence of the solutions to (2.34) requires some comments. The function $L'(\cdot)$ is singular and it keeps the value of z in the interval $[0,1]$. Thus, we approximate (2.34) by its regularization substituting $L'(\cdot)$ by $L'_\delta(\cdot)$ which is obtained in the same manner as for f_ϵ in (2.26). The estimates are the same, there is no problem to pass to the limit $\delta \to 0^+$, since we control the second derivatives of z. Hence, we ensure that $z \in [0,1]$ as well, which yields the proposition of the lemma. Concerning the continuity of $\mathcal{S}_2$, let us note that $t \mapsto K(t)t$ is Lipschitz continuous and $t \mapsto L'(t)$ is monotone, hence we can proceed similarly as in the case of $\mathcal{S}_1$. □

To conclude the proof of Proposition 2.9, we will use the Leray–Schauder fixed point theorem 4.9. We define the solution operator $\mathcal{T} : \mathbf{M}_p \to \mathbf{M}_p$, $\mathcal{T}(\mathbf{v}) = \mathbf{w}$ of the problem

$$\begin{aligned}-\mu\Delta\mathbf{w} - (\mu+\lambda)\nabla(\operatorname{div}\mathbf{w}) = -\frac{1}{2}\operatorname{div}(K(\varrho)\varrho\mathbf{v}\otimes\mathbf{v}) - \frac{1}{2}K(\varrho)\varrho\mathbf{v}\cdot\nabla\mathbf{v} - \nabla p_\epsilon(\varrho,c)\\ + K(\varrho)\varrho\mathbf{g} + \operatorname{div}\big(\nabla c\otimes\nabla c - \frac{|\nabla c|^2}{2}\mathbb{I}\big) - a\nabla c\int_0^\varrho sK'(s)\mathrm{d}s,\end{aligned} \tag{2.37}$$

where $\varrho = \mathcal{S}_1(\mathbf{v})$, and $c = \mathcal{S}_2(\mathbf{v})$, and equipped with the boundary condition

$$\mathbf{w}\cdot\mathbf{n} = 0, \quad \mathbf{n}\cdot\mathbb{T}(\mathbf{w})\cdot\boldsymbol{\tau}^k + \alpha\mathbf{w}\cdot\boldsymbol{\tau}^k = 0, \text{ on } \partial\Omega.$$

Lemma 2.12. *$\mathcal{T}$ is continuous and compact operator from $\mathbf{M}_p$ to $\mathbf{M}_p$ for $p > 3$.*

Proof. It is again one more time strictly elliptic system with right-hand side which belongs at least to the $L^p(\Omega)$, it contains at most the first order derivatives of $\varrho, \mathbf{v}$ and at most the second order derivatives of c, see e.g. [101, Lemma 3] for analogous considerations. □

Finally, we will verify that all possible solutions of $\ell\mathcal{T}(\mathbf{v}) = \mathbf{v}$, for $\ell \in [0,1]$ are bounded in $\mathbf{M}_p$ independently of ℓ. Testing the momentum equation by $\mathbf{v}$ yields (with use of the last equation tested by $\mathbf{v}\cdot\nabla c$)

$$\begin{aligned}\int_\Omega \ell\mathbf{v}\cdot\nabla p_\epsilon(\varrho,c) + \mu|\nabla\mathbf{v}|^2 + (\mu+\lambda)|\operatorname{div}\mathbf{v}|^2\,\mathrm{d}x + \int_{\partial\Omega}\alpha\big(|\mathbf{v}\cdot\boldsymbol{\tau}^1|^2 + |\mathbf{v}\cdot\boldsymbol{\tau}^2|^2\big)\mathrm{d}S\\ \leq \ell\int_\Omega K(\varrho)\varrho\mathbf{g}\cdot\mathbf{v}\,\mathrm{d}x + \ell\int_\Omega\Big(-K(\varrho)\varrho\overset{+}{c}\nabla c\cdot\mathbf{v} - K(\varrho)\varrho\frac{\partial f_\epsilon}{\partial c}\nabla c\cdot\mathbf{v}\Big)\,\mathrm{d}x\\ + \ell\int_\Omega\Big(-\epsilon\varrho K(\varrho)L'(c)\nabla c\cdot\mathbf{v} - a\nabla c\cdot\mathbf{v}\int_0^\varrho sK'(s)\mathrm{d}s\Big)\,\mathrm{d}x,\end{aligned} \tag{2.38}$$

from the equation for concentration we get $\int_\Omega \big(\overset{+}{c}\big)^2 \,\mathrm{d}x = \int_\Omega K(\varrho)\varrho\overset{+}{c}\mathbf{v}\cdot\nabla c\,\mathrm{d}x$. Next,

$$\begin{aligned}\int_\Omega \mathbf{v}\cdot\nabla p_\epsilon(\varrho,c)\ \mathrm{d}x &= \int_\Omega \mathbf{v}\cdot\nabla\varrho\Big(\gamma\varrho^{\gamma-1}K(\varrho)+(ac+d)K(\varrho)\frac{\mathrm{d}}{\mathrm{d}\varrho}\big(\varrho^2 l_\epsilon'(\varrho)\big)\Big)\ \mathrm{d}x\\ &\quad+\int_\Omega a\nabla c\cdot\mathbf{v}\int_0^\varrho K(s)\frac{\mathrm{d}}{\mathrm{d}s}\big(s^2 l_\epsilon'(s)\big)\mathrm{d}s\ \mathrm{d}x \end{aligned}\tag{2.39}$$

and

$$\begin{aligned}\int_\Omega K(\varrho)\varrho\mathbf{v}\cdot\nabla c\frac{\partial f_\epsilon}{\partial c}(\varrho,c)\,\mathrm{d}x &= \int_\Omega K(\varrho)\varrho\Big(\mathbf{v}\cdot\nabla\big(f_\epsilon+\varrho\frac{\partial f_\epsilon}{\partial\varrho}\big)-\varrho a l_\epsilon'(\varrho)\mathbf{v}\cdot\nabla c\Big)\,\mathrm{d}x\\ -\int_\Omega K(\varrho)\varrho\mathbf{v}\cdot\nabla\varrho\Big(2\Gamma'(\varrho)&+\varrho\Gamma''(\varrho)+(ac+d)l_\epsilon'(\varrho)+(ac+d)\frac{\mathrm{d}}{\mathrm{d}\varrho}\big(\varrho l_\epsilon'(\varrho)\big)\Big)\,\mathrm{d}x\\ &= \int_\Omega \mathrm{div}\Big(K(\varrho)\varrho\mathbf{v}\big(f_\epsilon+\varrho\frac{\partial f_\epsilon}{\partial\varrho}\big)\Big)-\mathrm{div}\big(K(\varrho)\varrho\mathbf{v}\big)\big(f_\epsilon+\varrho\frac{\partial f_\epsilon}{\partial\varrho}\big)\,\mathrm{d}x\\ &\quad-\int_\Omega a\nabla c\cdot\mathbf{v}\int_0^\varrho\frac{\mathrm{d}}{\mathrm{d}s}\big(K(s)s^2 l_\epsilon'(s)\big)\mathrm{d}s\ \mathrm{d}x\\ &\quad+\int_\Omega K(\varrho)\mathbf{v}\cdot\nabla\varrho\frac{\mathrm{d}}{\mathrm{d}\varrho}\big(\varrho^2\Gamma'(\varrho)+\varrho^2 l_\epsilon'(\varrho)(ac+d)\big)\,\mathrm{d}x.\end{aligned}\tag{2.40}$$

The first term on the right-hand side of (2.40) can be eliminated by boundary conditions, for the second one we will use the continuity equation (tested by quantity $f_\epsilon+\varrho\frac{\partial f_\epsilon}{\partial\varrho}$). The second integral simply cancels out with the last terms of relations (2.38) and (2.39), and the last term is exactly the same as the main part of (2.39).

Thus, summing up the resulting inequalities with appropriate powers of ℓ we get

$$\begin{aligned}\|\mathbf{v}\|^2_{W^{1,2}(\Omega)}+\ell\big\|\overset{+}{c}\big\|_2^2+\ell\int_\Omega\epsilon\varrho\big(f_\epsilon+\varrho\frac{\partial f_\epsilon}{\partial\varrho}\big)\,\mathrm{d}x&\\ \le\ell\int_\Omega K(\varrho)\varrho\mathbf{g}\cdot\mathbf{v}\,\mathrm{d}x-\ell\int_{\Omega_K}\epsilon\varrho K(\varrho)L'(c)\mathbf{v}\cdot\nabla c\,\mathrm{d}x&\\ +\ell\int_\Omega\big(\epsilon\Delta\varrho+\epsilon K(\varrho)h\big)\big(f_\epsilon+\varrho\frac{\partial f_\epsilon}{\partial\varrho}\big)\,\mathrm{d}x,&\end{aligned}\tag{2.41}$$

where we have introduced the notation $\Omega_K=\{K(\varrho)\varrho>0\}\cap\Omega$. We estimate the ϵ-dependent terms

$$\begin{aligned}-\epsilon\int_{\Omega_K}K(\varrho)\varrho L'(c)\nabla c\cdot\mathbf{v}\,\mathrm{d}x&=-\int_{\Omega_K}\epsilon K(\varrho)\varrho\mathbf{v}\cdot\nabla\big(L(c)\big)\,\mathrm{d}x\\ &=\int_{\partial\Omega_K}\epsilon\varrho K(\varrho)\mathbf{v}\cdot\mathbf{n}L(c)\mathrm{d}S+\int_{\Omega_K}\epsilon\,\mathrm{div}(K(\varrho)\varrho\mathbf{v})L(c)\,\mathrm{d}x\\ &\le\sqrt{\epsilon}\,\|\nabla\varrho\|_{L^2(\Omega)}\,\|\varrho K'(\varrho)+K(\varrho)\|_{L^3(\Omega_K)}\,\|\mathbf{v}\|_{L^6(\Omega)}\,\sqrt[4]{\epsilon}\,\|L(c)\|_{L^\infty(\Omega_K)}\,\sqrt[4]{\epsilon}\\ &\quad+\|K(\varrho)\varrho\|_{L^2(\Omega)}\,\|\nabla\mathbf{v}\|_{L^2(\Omega)}\,\sqrt[4]{\epsilon}\,\|L(c)\|_{L^\infty(\Omega_K)}\,\epsilon^{3/4}\end{aligned}$$

and we can use $\sqrt[4]{\epsilon}\,\|L(c)\|_{L^\infty(\Omega_K)}\le C(\sqrt[4]{\epsilon}\,\big\|(K(\varrho)\varrho)^{1/(\beta+1)}L'(c)\big\|^{1/4}_{L^{\beta+1}(\Omega)}+1)$ with estimate (2.36).

Concerning the term $\epsilon K(\varrho)(f_\epsilon + \varrho\frac{\partial f_\epsilon}{\partial \varrho})$ on the right-hand side of (2.41) we can decompose it into three parts using the structure of f_ϵ. First, $\Gamma_\epsilon(\varrho) + \varrho\Gamma'_\epsilon(\varrho)$, which can be bounded by the corresponding term on the left-hand side, see $\epsilon(\varrho f_\epsilon)$. Second, $(ac+d)\big(1+l_\epsilon(\varrho)\big)$, which has good sign on the set $\{\varrho \le \mathrm{e}^{-1}\}$ and is bounded by $c(1+\varrho)K(\varrho)$ on $\{\varrho > \mathrm{e}^{-1}\}$. The third term $\epsilon\big(L_\epsilon(c) + b(c)\big)$ can be bounded according to our definition of l_ϵ. For $\Delta\varrho(f_\epsilon + \varrho\frac{\partial f_\epsilon}{\partial\varrho})$ we have

$$\int_\Omega \epsilon\Delta\varrho(f_\epsilon + \varrho\frac{\partial f_\epsilon}{\partial \varrho})\,\mathrm{d}x = -\epsilon\int_\Omega |\nabla\varrho|^2\left(\gamma\varrho^{\gamma-2} + (ac+d)\big(\varrho l''_\epsilon(\varrho) + 2l'_\epsilon(\varrho)\big)\right)\mathrm{d}x \\ - \epsilon\int_\Omega \nabla\varrho\cdot\nabla c\big(l'_\epsilon(\varrho)\varrho + b'(c) + L'_\epsilon(c)\big)\,\mathrm{d}x. \quad (2.42)$$

The first integral on the right-hand side of inequality (2.42) has a good sign. Indeed, $\varrho^{\gamma-2}$ as well as $\varrho l''_\epsilon(\varrho) + 2l'_\epsilon(\varrho)$ are non-negative (for large arguments we have $\varrho l''_\epsilon(\varrho) + 2l'_\epsilon(\varrho) = \frac{1}{\varrho} \ge 0$, in the other case the conclusion is obtained from the fact that for small arguments l_ϵ is increasing and convex, see (2.26)). Note also that we have (2.33) and for $\epsilon > 0$ the approximated version of (2.4) yields $c \in [0,1]$ as soon as we control $\|K(\varrho)\varrho\|_{L^{2\gamma}}\left\|\overset{+}{c}\right\|_{L^2}$. For the rest we get denoting $V_\epsilon(c) = b'(c) + L'_\epsilon(c)$ and $U = \|\mathbf{v}\|^2_{W^{1,2}(\Omega)} + \ell\left\|\overset{+}{c}\right\|^2_{L^2(\Omega)}$ that

$$\begin{aligned}\ell\int_\Omega \epsilon\,|\nabla\varrho|\,|\nabla c|\,|l'_\epsilon(\varrho)\varrho + V_\epsilon(c)|\,\mathrm{d}x \\ &\le \ell\sqrt[4]{\epsilon}\left\|\sqrt{\epsilon}\nabla\varrho\right\|_{L^2(\Omega)}\|\nabla c\|_{L^2(\Omega)}\sqrt[4]{\epsilon}\big(\|l'_\epsilon(\varrho)\varrho + V_\epsilon(c)\|_{L^\infty(\Omega)}\big) \\ &\le \ell\sqrt[4]{\epsilon}C\left\|K(\varrho)\varrho^2\right\|^{1/2}_{L^2(\Omega)}\|\nabla\mathbf{v}\|^{1/2}_{L^2(\Omega)}\|K(\varrho)\varrho\|_{L^\gamma(\Omega)}\left\|\overset{+}{c}\right\|_2 \\ &\le \ell\sqrt[4]{\epsilon}C\left\|K(\varrho)\varrho^2\right\|^{1/2}_{L^2(\Omega)}\|K(\varrho)\varrho\|_{L^\gamma(\Omega)}U^{3/4},\end{aligned} \quad (2.43)$$

where we have used the choice of l_ϵ which guarantees that $\sqrt[4]{\epsilon}\,\|l'_\epsilon(\varrho)\varrho\|_{L^\infty(\Omega)} + \sqrt[4]{\epsilon}\,\|V_\epsilon(c)\|_{L^\infty(\Omega)}\big) \le C$ independently of ϵ. Thus,

$$U \le C\left(1 + \|K(\varrho)\varrho\|^2_{L^{6/5}} + \epsilon\,\|K(\varrho)\varrho\|_{L^2}U^{3/4} + \sqrt[4]{\epsilon}\left\|K(\varrho)\varrho^2\right\|^{1/2}_{L^2}\|K(\varrho)\varrho\|_{L^3}U^{3/4}\right).$$

However $\|K(\varrho)\varrho\|_{L^6(\Omega)}$ is finite, so finally,

$$U \le C\Big(1 + \|K(\varrho)\varrho\|^2_{L^{6/5}(\Omega)}\Big), \qquad \|\nabla c\|_{L^{\frac{6q}{6+q}}(\Omega)} \le C(\|K(\varrho)\varrho\|_{L^q(\Omega)}\left\|\overset{+}{c}\right\|_2 + 1).$$

The Bogovskii estimates go along exactly the same lines as in the a priori approach as soon as we observe that

$$\ell\int_\Omega\left|a\mathbf{\Phi}\cdot\nabla c\int_0^\varrho tK'(t)\mathrm{d}t\right|\mathrm{d}x \le C\ell\,\|\nabla c\|_{L^{\frac{6\gamma}{3+\gamma}}(\Omega)}\|K(\varrho)\varrho\|_{L^{\frac{6\gamma}{3\gamma-3}}(\Omega)}\|\mathbf{\Phi}\|_{L^3(\Omega)} \\ \le C\Big(\|K(\varrho)\varrho\|^{2+2(\gamma-3)}_{L^{2\gamma}(\Omega)} + 1\Big),$$

so consequently

$$\|\varrho_\epsilon\|_{L^{3\gamma-6}(\Omega)} \le C,$$

independently of k and ϵ. Furthermore, by the same iteration process applied on the last two equations of (2.1)–(2.4) as in the a priori approach, we can deduce that

$$\|\overset{+}{c}_\epsilon\|_{L^{\frac{6\gamma}{\gamma+3}}(\Omega)} + \|\nabla c_\epsilon\|_{L^\infty(\Omega)} + \|\Delta c_\epsilon\|_{L^{\frac{6\gamma}{6+\gamma}}(\Omega)} \leq C. \tag{2.44}$$

Having these estimates in hands and noting that $\|K(\varrho)\varrho\|_{L^\infty(\Omega)} \leq C(k)$ we can apply the elliptic theory on the equation (2.37) and get the estimate of fixed points of $\mathcal{T}$ in $\mathbf{M}_p$ for $p = \frac{6\gamma}{6+\gamma}$. This completes the proof of Proposition 2.9. $\square$

2.4 Artificial diffusion limit

This section is dedicated to the proof of convergence of the constructed approximative solutions to a weak solution to the original system. As usually the key part is related to the proof of the strong convergence of the densities.

Thanks to the estimates (2.32), (2.44) we extract from the family $(\varrho_\epsilon, \mathbf{v}_\epsilon, \overset{+}{c}_\epsilon, c_\epsilon)$ subsequences which converge in the corresponding spaces as $\epsilon \to 0^+$. Namely,[10]

$$\begin{aligned}
&\mathbf{v}_\epsilon \rightharpoonup \mathbf{v} \text{ in } W^{1,q}(\Omega), && \mathbf{v}_\epsilon \to \mathbf{v} \text{ in } L^\infty(\Omega),\\
&\varrho_\epsilon \rightharpoonup^* \varrho \text{ in } L^\infty(\Omega), && p_\epsilon(\varrho, c) \rightharpoonup^* \overline{p(\varrho, c)} \text{ in } L^\infty(\Omega),\\
&K(\varrho_\epsilon)\varrho_\epsilon \rightharpoonup^* \overline{K(\varrho)\varrho} \text{ in } L^\infty(\Omega), && K(\varrho_\epsilon) \rightharpoonup^* \overline{K(\varrho)} \text{ in } L^\infty(\Omega),\\
&\int_0^{\varrho_\epsilon} tK'(t)\mathrm{d}t \rightharpoonup^* \overline{\int_0^{\varrho} tK'(t)\mathrm{d}t} \text{ in } L^\infty(\Omega), && h_\epsilon(\varrho_\epsilon) \rightharpoonup^* \overline{h_\epsilon(\varrho)} \text{ in } L^\infty(\Omega),\\
&c_\epsilon \rightharpoonup c \text{ in } W^{2,2}(\Omega), && \nabla c_\epsilon \to \nabla c \text{ in } L^6(\Omega),\\
&\epsilon K(\varrho_\epsilon)\varrho_\epsilon L'(c_\epsilon) \rightharpoonup \overline{\epsilon K(\varrho)\varrho} L'(c) \text{ in } L^2(\Omega), && \overset{+}{c}_\epsilon \rightharpoonup \overset{+}{c} \text{ in } L^2(\Omega),\\
&K(\varrho_\epsilon)\varrho_\epsilon L'_\epsilon(c_\epsilon) \rightharpoonup \overline{K(\varrho)\varrho} L'(c) \text{ in } L^{12/7}(\Omega), && K(\varrho_\epsilon)\varrho_\epsilon \overset{+}{c}_\epsilon \rightharpoonup \overline{K(\varrho)\varrho\overset{+}{c}} \text{ in } L^2(\Omega).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\operatorname{div}(\overline{K(\varrho)\varrho}\mathbf{v}) = 0,\\
&\frac{1}{2}\operatorname{div}(\overline{K(\varrho)\varrho}\mathbf{v}\otimes\mathbf{v}) + \frac{1}{2}\overline{K(\varrho)\varrho}\,\mathbf{v}\cdot\nabla\mathbf{v} - \mu\Delta\mathbf{v} - (\mu+\lambda)\nabla(\operatorname{div}\mathbf{v}) + \nabla\overline{p_\epsilon(\varrho, c)}\\
&\qquad = \overline{K(\varrho)\varrho}\mathbf{g} + \operatorname{div}\big(\nabla c\otimes\nabla c - \frac{|\nabla c|^2}{2}\mathbb{I}\big) - a\nabla c\overline{\int_0^\varrho tK'(t)\mathrm{d}t},\\
&\overline{K(\varrho)\varrho}\,\mathbf{v}\cdot\nabla c = \overset{+}{c},\\
&\qquad \overline{K(\varrho)\varrho\overset{+}{c}} = \Delta c - \overline{h_\epsilon(\varrho)} - \overline{K(\varrho)\varrho}\big(L'(c) + b'(c)\big) - \overline{\epsilon K(\varrho)\varrho} L'(c) \text{ in } \Omega,\\
&\qquad \mathbf{v}\cdot\mathbf{n} = 0, \qquad \nabla c\cdot\mathbf{n} = 0,\\
&\qquad \mathbf{n}\cdot\mathbb{T}\cdot\boldsymbol{\tau}^k + \alpha\mathbf{v}\cdot\boldsymbol{\tau}^k = 0, \text{ on } \partial\Omega,
\end{aligned}$$

where $h_\epsilon(s) = as\cdot l_\epsilon(s)$. Note that, due to the high regularity we have the pointwise convergence of concentrations. In order to show the pointwise convergence of densities we need to investigate the momentum equation, especially its potential part defining the effective viscous flux. Let us decompose the velocity field $\mathbf{v}$ using Helmholtz decomposition from Theorem 4.19, id est

$$\mathbf{v} = P_H\mathbf{v} + \nabla P_\nabla\mathbf{v},$$

[10] Recall that we denote a weak limit of nonlinear expressions $\{g_\epsilon(\varrho_\epsilon, \mathbf{v}_\epsilon, \overset{+}{c}_\epsilon, c_\epsilon)\}$ by $\overline{g(\varrho, \mathbf{v}, \overset{+}{c}, c)}$.

where

$$\begin{aligned} \operatorname{curl} P_H \mathbf{v} = \operatorname{curl} \mathbf{v} = \boldsymbol{\omega} \text{ in } \Omega, \\ \operatorname{div} P_H \mathbf{v} = 0 \text{ in } \Omega, \\ P_H \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega, \end{aligned} \qquad \text{and} \qquad \begin{aligned} \Delta P_\nabla \mathbf{v} = \operatorname{div} \mathbf{v} \text{ in } \Omega, \\ \nabla P_\nabla \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega, \\ \int_\Omega P_\nabla \mathbf{v} \, \mathrm{d}x = 0. \end{aligned}$$

For the solenoidal part $P_H \mathbf{v}$ we have good estimates, see Theorem 4.14

$$\|\nabla P_H \mathbf{v}\|_{L^q(\Omega)} \leq \|\boldsymbol{\omega}\|_{L^q(\Omega)}, \qquad \|\nabla^2 P_H \mathbf{v}\|_{L^q(\Omega)} \leq \|\boldsymbol{\omega}\|_{W^{1,q}(\Omega)},$$

and since $\boldsymbol{\omega}$ solves with corresponding boundary conditions equation

$$-\mu \Delta \boldsymbol{\omega} = -\operatorname{curl}\big(\overline{K(\varrho)\varrho}\mathbf{v} \cdot \nabla \mathbf{v}\big) + \operatorname{curl}\Big(\overline{K(\varrho)\varrho}\mathbf{g} + \Delta c \nabla c - a \nabla c \overline{\int_0^\varrho tK'(t)\mathrm{d}t}\Big), \tag{2.45}$$

we have $\|\boldsymbol{\omega}\|_{W^{1,q}(\Omega)} \leq C(k)$. Similarly we also decompose the approximative velocity field as $\mathbf{v}_\epsilon = \nabla P_\nabla \mathbf{v}_\epsilon + P_H \mathbf{v}_\epsilon$ and deduce due to the slip boundary conditions the following problem for vorticity

$$\begin{aligned} -\mu \Delta \boldsymbol{\omega}_\epsilon = & \underbrace{\operatorname{curl}\Big(K(\varrho_\epsilon)\varrho_\epsilon \mathbf{g} - K(\varrho_\epsilon)\varrho_\epsilon \mathbf{v}_\epsilon \cdot \nabla \mathbf{v}_\epsilon - \frac{1}{2}\epsilon m K(\varrho_\epsilon)\mathbf{v}_\epsilon + \epsilon \frac{1}{2} \varrho_\epsilon \mathbf{v}_\epsilon + \Delta c_\epsilon \nabla c_\epsilon - a \nabla c_\epsilon \int_0^{\varrho_\epsilon} tK'(t)\mathrm{d}t\Big)}_{=:\mathbf{H}_1} \underbrace{- \operatorname{curl}\big(\frac{1}{2}\epsilon \Delta \varrho_\epsilon \mathbf{v}_\epsilon\big)}_{=:\mathbf{H}_2}, \\ & \boldsymbol{\omega}_\epsilon \cdot \boldsymbol{\tau}^1 = -(2\chi_2 - \frac{\alpha}{\mu})\mathbf{v}_\epsilon \cdot \boldsymbol{\tau}^2 \text{ on } \partial\Omega, \\ & \boldsymbol{\omega}_\epsilon \cdot \boldsymbol{\tau}^2 = (2\chi_1 - \frac{\alpha}{\mu})\mathbf{v}_\epsilon \cdot \boldsymbol{\tau}^1 \text{ on } \partial\Omega, \\ & \operatorname{div} \boldsymbol{\omega}_\epsilon = 0 \text{ on } \partial\Omega. \end{aligned}$$

The structure of the right-hand side of the equation enables us to consider $\boldsymbol{\omega}_\epsilon = \boldsymbol{\omega}_\epsilon^0 + \boldsymbol{\omega}_\epsilon^1 + \boldsymbol{\omega}_\epsilon^2$ as a sum of solutions to three particular systems, namely

$$\begin{aligned} &\mu \Delta \boldsymbol{\omega}_\epsilon^0 = \mathbf{0}, && \mu \Delta \boldsymbol{\omega}_\epsilon^1 = \mathbf{H}_1, && \mu \Delta \boldsymbol{\omega}_\epsilon^2 = \mathbf{H}_2 \text{ in } \Omega, \\ &\boldsymbol{\omega}_\epsilon^0 \cdot \boldsymbol{\tau}^1 = -(2\chi_2 - \frac{\alpha}{\mu})\mathbf{v}_\epsilon \cdot \boldsymbol{\tau}^2, && \boldsymbol{\omega}_\epsilon^1 \cdot \boldsymbol{\tau}^1 = 0, && \boldsymbol{\omega}_\epsilon^2 \cdot \boldsymbol{\tau}^1 = 0 \text{ on } \partial\Omega, \\ &\boldsymbol{\omega}_\epsilon^0 \cdot \boldsymbol{\tau}^2 = -(2\chi_1 - \frac{\alpha}{\mu})\mathbf{v}_\epsilon \cdot \boldsymbol{\tau}^1, && \boldsymbol{\omega}_\epsilon^1 \cdot \boldsymbol{\tau}^2 = 0, && \boldsymbol{\omega}_\epsilon^2 \cdot \boldsymbol{\tau}^2 = 0 \text{ on } \partial\Omega, \\ &\operatorname{div} \boldsymbol{\omega}_\epsilon^0 = 0, && \operatorname{div} \boldsymbol{\omega}_\epsilon^1 = 0, && \operatorname{div} \boldsymbol{\omega}_\epsilon^2 = 0 \text{ on } \partial\Omega. \end{aligned} \tag{2.46}$$

Lemma 2.13. *If the vorticity $\boldsymbol{\omega}_\epsilon = \boldsymbol{\omega}_\epsilon^0 + \boldsymbol{\omega}_\epsilon^1 + \boldsymbol{\omega}_\epsilon^2$ solves (2.46), then we have*

$$\|\boldsymbol{\omega}_\epsilon^0\|_{W^{1,q}(\Omega)} + \|\boldsymbol{\omega}_\epsilon^1\|_{W^{1,q}(\Omega)} \leq C\big(1 + k^{1+\gamma(\frac{4}{3}-\frac{2}{q})}\big), \quad \text{for } q \in \Big[2, \frac{6\gamma}{6+\gamma}\Big], \tag{2.47}$$

$$\|\boldsymbol{\omega}_\epsilon^2\|_{L^q(\Omega)} \leq C(k)\epsilon^{1/2}, \quad \text{for } q \in [1,2]. \tag{2.48}$$

Proof. Following closely the corresponding considerations in [132], we deduce that

$$\|\boldsymbol{\omega}_\epsilon^1\|_{W^{1,q}(\Omega)} \leq C\|\mathbf{H}_1\|_{W^{-1,q}(\Omega)} \text{ and } \|\boldsymbol{\omega}_\epsilon^0\|_{W^{1,q}(\Omega)} \leq C\|\mathbf{v}_\epsilon\|_{W^{1,q}(\Omega)},$$

but according to (2.44) for any q such that $2 \le q \le \frac{6\gamma}{6+\gamma}$ we have

$$\|\mathbf{v}_\epsilon\|_{W^{1,q}(\Omega)} \le C(\|p_\epsilon(\varrho_\epsilon, c_\epsilon)\|_{L^q(\Omega)} + \|\Delta c_\epsilon \nabla c_\epsilon\|_{L^q(\Omega)} + 1) \le C\big(1 + k^{\gamma(1-\frac{2}{q})}\big),$$

so we concentrate on $\boldsymbol{\omega}_\epsilon^1$. If we denote $r = \frac{6\gamma}{6+\gamma} > 3$, we get by interpolation

$$\begin{aligned}
\|\boldsymbol{\omega}_\epsilon^1\|_{W^{1,q}(\Omega)} \le& C\big(1 + \|K(\varrho_\epsilon)\varrho_\epsilon \mathbf{v}_\epsilon \nabla \mathbf{v}_\epsilon\|_{L^q(\Omega)} + \|\Delta c_\epsilon \nabla c_\epsilon\|_{L^q(\Omega)}\big) \\
\le& C\big(1 + k\,\|\mathbf{v}_\epsilon\|_{L^\infty(\Omega)} \|\nabla \mathbf{v}_\epsilon\|_{L^q(\Omega)}\big) \\
\le& C\big(1 + k\,\|\mathbf{v}_\epsilon\|_{L^6(\Omega)}^{\frac{2r-6}{3r-6}} \|\nabla \mathbf{v}_\epsilon\|_{L^r(\Omega)}^{\frac{r}{3r-6}} \|\nabla \mathbf{v}_\epsilon\|_{L^2(\Omega)}^{\frac{2(r-q)}{q(r-2)}} \|\nabla \mathbf{v}_\epsilon\|_{L^r(\Omega)}^{\frac{r(q-2)}{q(r-2)}}\big) \qquad (2.49)\\
\le& C\big(1 + k\,\|\nabla \mathbf{v}_\epsilon\|_{L^r(\Omega)}^{\frac{r}{3r-6}+\frac{r(q-2)}{q(r-2)}}\big) \le C\big(1 + k^{1+\frac{4rq-6r}{3q(r-2)}\gamma(1-\frac{2}{r})}\big) \\
& \le C\big(1 + k^{1+\gamma(\frac{4}{3}-\frac{2}{q})}\big).
\end{aligned}$$

Finally, for the last part we get for $q \le 2$

$$\begin{aligned}
\|\boldsymbol{\omega}_\epsilon^2\|_{L^q(\Omega)} \le& C\,\|\epsilon \Delta \varrho_\epsilon \mathbf{v}_\epsilon\|_{W^{-1,q}(\Omega)} \\
& \le C\epsilon\big(\|\nabla \varrho_\epsilon \mathbf{v}_\epsilon\|_{L^q(\Omega)} + \|\nabla \varrho_\epsilon \nabla \mathbf{v}_\epsilon\|_{L^{\frac{6}{5}}(\Omega)}\big) \le C(k)\epsilon^{1/2}. \quad \square
\end{aligned}$$

Now, we are approaching the key definition of the effective viscous flux. Inserting the Helmholtz decomposition into the approximative momentum equation yields

$$\begin{aligned}
\nabla G_\epsilon =& \mu \Delta P_H \mathbf{v}_\epsilon + K(\varrho_\epsilon)\varrho_\epsilon \mathbf{g} - K(\varrho_\epsilon)\varrho_\epsilon \mathbf{v}_\epsilon \cdot \nabla \mathbf{v}_\epsilon - \frac{1}{2}\epsilon \Delta \varrho_\epsilon \mathbf{v}_\epsilon - \frac{1}{2}\epsilon m K(\varrho_\epsilon)\mathbf{v}_\epsilon \\
& + \epsilon \frac{1}{2}\varrho_\epsilon \mathbf{v}_\epsilon + \operatorname{div}\big(\nabla c_\epsilon \otimes \nabla c_\epsilon - \frac{|\nabla c_\epsilon|^2}{2}\mathbb{I}\big) - a \nabla c_\epsilon \int_0^{\varrho_\epsilon} t K'(t) \mathrm{d}t, \quad (2.50)
\end{aligned}$$

where we have introduced the fundamental quantity

$$G_\epsilon = -(2\mu + \lambda)\Delta P_\nabla \mathbf{v}_\epsilon + p_\epsilon(\varrho_\epsilon, c_\epsilon) = -(2\mu + \lambda) \operatorname{div} \mathbf{v}_\epsilon + p_\epsilon(\varrho_\epsilon, c_\epsilon). \qquad (2.51)$$

Similarly, inserting the Helmholtz decomposition into the limit momentum equation we obtain (with use of the fact that due to the continuity equation we have $\mathbf{v} \cdot \overline{K(\varrho)\varrho \nabla \mathbf{v}} = \overline{K(\varrho)\varrho}\mathbf{v} \cdot \nabla \mathbf{v}$) that

$$\begin{aligned}
\nabla\big(-(2\mu + \lambda)\Delta P_\nabla \mathbf{v} + \overline{p(\varrho, c)}\,\big) =& \mu \Delta P_H \mathbf{v} + \overline{K(\varrho)\varrho}\mathbf{g} - \overline{K(\varrho)\varrho}\mathbf{v} \cdot \nabla \mathbf{v} \\
& + \operatorname{div}\big(\nabla c \otimes \nabla c - \frac{|\nabla c|^2}{2}\mathbb{I}\big) - a \nabla c \overline{\int_0^{\varrho} t K'(t) \mathrm{d}t}, \quad (2.52)
\end{aligned}$$

hence we define the limit version of effective viscous flux by

$$G = -(2\mu + \lambda)\Delta P_\nabla \mathbf{v} + \overline{p(\varrho, c)} = -(2\mu + \lambda) \operatorname{div} \mathbf{v} + \overline{p(\varrho, c)}. \qquad (2.53)$$

Note we have control of $\int_\Omega G_\epsilon \,\mathrm{d}x = \int_\Omega p_\epsilon(\varrho_\epsilon, c_\epsilon)\ \mathrm{d}x$ and $\int_\Omega G \,\mathrm{d}x = \int_\Omega \overline{p(\varrho, c)}\ \mathrm{d}x$. Further we state the most important features of the effective viscous flux.

Lemma 2.14. *There exists a subsequence such that*

$$G_\epsilon \to G \text{ (strongly) in } L^2(\Omega),$$

and

$$\|G\|_{L^\infty(\Omega)} \le C(\zeta)\big(1 + k^{1+\frac{2\gamma}{3}+\zeta}\big), \text{ for any } \zeta \in \Big(0, \frac{\gamma - 6}{3}\Big]. \qquad (2.54)$$

Proof. Let us decompose G_ϵ to $G_\epsilon = G_\epsilon^1 + G_\epsilon^2$, where G_ϵ^2 contains the "strongly ϵ-dependent" terms of the right-hand side of (2.51), namely $\nabla G_\epsilon^2 = -\epsilon \frac{1}{2}\Delta\varrho_\epsilon \mathbf{v}_\epsilon - \mu \operatorname{curl} \boldsymbol{\omega}_\epsilon^2$ with $\int_\Omega G_\epsilon^2 \,\mathrm{d}x = 0$, so

$$\left\|G_\epsilon^2\right\|_{L^q(\Omega)} \le C(\epsilon \left\|\Delta\varrho_\epsilon \mathbf{v}_\epsilon\right\|_{W^{-1,q}(\Omega)} + \mu \left\|\operatorname{curl} \boldsymbol{\omega}_\epsilon^2\right\|_{W^{-1,q}(\Omega)}) \le C(k)\epsilon^{1/2}, \text{ for } q \in [1,2].$$

Using once more the estimates from Lemma 2.13, we observe that $\left|\int_\Omega G_\epsilon \,\mathrm{d}x\right| \le C$ and

$$\left\|G_\epsilon^1\right\|_{W^{1,q}(\Omega)} \le C\big(1 + k^{1+\gamma(\frac{4}{3}-\frac{2}{q})}\big), \text{ for } q \in \left[2, \frac{6\gamma}{6+\gamma}\right]. \tag{2.55}$$

Therefore, since $\gamma > 6$ we have, at least for a suitably chosen subsequence

$$G_\epsilon^1 \to G^1 \text{ (strongly) in } L^\infty(\Omega) \text{ and } G_\epsilon^2 \to 0 \text{ (strongly) in } L^2(\Omega).$$

Thus,

$$G_\epsilon = G_\epsilon^1 + G_\epsilon^2 \to G^1 = G \text{ (strongly) in } L^q(\Omega), \text{ for } q \in [1,2].$$

Finally, setting $q = 3 + \frac{3\zeta}{2\gamma - 3\zeta}$ in (2.55) we get the desired conclusion

$$\|G\|_{L^\infty(\Omega)} \le C(q)\|G\|_{W^{1,q}(\Omega)} \le C(\zeta)\big(1 + k^{1+\frac{2\gamma}{3}+\zeta}\big). \qquad \square$$

Now, we are ready to show that we are able to choose k in such a way that actually $\overline{K(\varrho)\varrho} = \varrho$ a.e. in Ω. This will be an immediate consequence of the following lemma.

Lemma 2.15. *There exists k_0 such that*

$$\frac{k-3}{k}(k-3)^\gamma - \|G\|_{L^\infty(\Omega)} \ge 1 \textit{ for } k > k_0, \tag{2.56}$$

and at least for a subsequence

$$\lim_{\epsilon \to 0} |\{\varrho_\epsilon > k-3\}| = 0. \tag{2.57}$$

Proof. Let us define a smooth non-increasing function $N : [0,\infty) \to [0,1]$ such that

$$N(t) = \begin{cases} 1 & \text{for } t \in [0, k-3], \\ \in [0,1] & \text{for } t \in (k-3, k-2), \\ 0 & \text{for } t \in [k-2, \infty), \end{cases}$$

and multiply the approximative continuity equation by $N^l(\varrho_\epsilon)$, for some suitable power $l \in \mathbb{N}$ in order to get after some manipulations

$$\int_\Omega \left(\int_0^{\varrho_\epsilon} \big(tlN^{l-1}(t)N'(t)\big)\mathrm{d}t \operatorname{div} \mathbf{v}_\epsilon \right) \mathrm{d}x \ge R_\epsilon \tag{2.58}$$

with $R_\epsilon = \epsilon \int_\Omega N^l(\varrho_\epsilon)\Delta\varrho_\epsilon + (m - \varrho_\epsilon)N^l(\varrho_\epsilon)\,\mathrm{d}x$, $R_\epsilon \to 0$ as $\epsilon \to 0$.[11] Further by definition of G_ϵ we have

$$\begin{aligned} &-(k-3)\int_\Omega \left(\int_0^{\varrho_\epsilon} lN^{l-1}(t)N'(t)\mathrm{d}t \right) p_\epsilon(\varrho_\epsilon, c_\epsilon)\,\mathrm{d}x \\ &\qquad \le k\left| \int_\Omega \left(\int_0^{\varrho_\epsilon} -lN^{l-1}(t)N'(t)\mathrm{d}t \right) G_\epsilon \,\mathrm{d}x \right| + R_\epsilon, \end{aligned} \tag{2.59}$$

[11] Note that $-\epsilon l \int_\Omega N^{l-1}(\varrho_\epsilon)N'(\varrho_\epsilon)\left|\nabla\varrho_\epsilon\right|^2 \mathrm{d}x \ge 0$

and

$$\frac{k-3}{k} \int\limits_{\{\varrho_\epsilon > k-3\}} \big(1 - N^l(\varrho_\epsilon)\big) p_\epsilon(\varrho_\epsilon, c_\epsilon)\,\mathrm{d}x \le \int\limits_{\{\varrho_\epsilon > k-3\}} \big(1 - N^l(\varrho_\epsilon)\big)\,|G_\epsilon|\,\mathrm{d}x + |R_\epsilon|\,.$$

Recalling the structure of the pressure we have according to (2.26) and the fact that $c_\epsilon \in [0,1]$

$$p_\epsilon(\varrho_\epsilon, c_\epsilon) = P_b(\varrho_\epsilon) + (ac_\epsilon + d) \int_0^{\varrho_\epsilon} K(t)\partial_t\big(t^2 l'_\epsilon(t)\big)\,\mathrm{d}t \ge P_b(\varrho_\epsilon), \tag{2.60}$$

which yields

$$\begin{aligned} \frac{k-3}{k}(k-3)^\gamma\,|\{\varrho_\epsilon > k-3\}| - \frac{k-3}{k}\,\|p_\epsilon(\varrho_\epsilon, c_\epsilon)\|_{L^2(\Omega)}\,\big\|N^l(\varrho_\epsilon)\big\|_{L^2(\Omega)} \\ \le \|G\|_{L^\infty(\Omega)}\,|\{\varrho_\epsilon > k-3\}| + \|G - G_\epsilon\|_{L^1(\Omega)} + |R_\epsilon|\,. \end{aligned} \tag{2.61}$$

According to (2.54) we are able to choose k_0 satisfying (2.56), yielding

$$|\{\varrho_\epsilon > k-3\}| \le C\Big(\big\|N^l(\varrho_\epsilon)\big\|_{L^2(\{\varrho_\epsilon > k-3\})} + \|G - G_\epsilon\|_{L^1(\Omega)} + |R_\epsilon|\Big). \tag{2.62}$$

However, the last two terms tend to zero as $\epsilon \to 0^+$ and as soon as we fix $\epsilon > 0$ we have $\big\|N^l(\varrho_\epsilon)\big\|_{L^2(\{\varrho_\epsilon > k-3\})} \to 0$ as $l \to +\infty$ as well. Thus, Lemma 2.15 is proved. □

Finally, we deduce the pointwise convergence of the densities. Our main aim is to show that

$$\overline{P_b(\varrho)\varrho} = \overline{P_b(\varrho)}\varrho,$$

which will further lead to $\varrho_\epsilon \to \varrho$ strongly in $L^q(\Omega)$ for $q < \infty$.

Lemma 2.16. *The weak limits satisfy*

$$\int_\Omega \overline{p(\varrho, c)\varrho}\,\mathrm{d}x \le \int_\Omega G\varrho\,\mathrm{d}x. \tag{2.63}$$

Proof. We test the approximative continuity equation by $\log k - \log(\varrho_\epsilon + \delta)$ and then consider the limit for $\delta \to 0^+$, see [100]. Since

$$\int_\Omega \epsilon\Delta\varrho_\epsilon\big(\log k - \log(\varrho_\epsilon + \delta)\big)\,\mathrm{d}x = \epsilon \int_\Omega \frac{1}{\varrho_\epsilon + \delta}\,|\nabla\varrho_\epsilon|^2\,\mathrm{d}x \ge 0,$$

we obtain

$$\int_\Omega \Big(\mathrm{div}\big(K(\varrho_\epsilon)\varrho_\epsilon\mathbf{v}_\epsilon\big) + \epsilon\varrho_\epsilon - \epsilon m K(\varrho_\epsilon)\Big)\big(\log k - \log(\varrho_\epsilon + \delta)\big)\,\mathrm{d}x \ge 0,$$

$$\int_\Omega K(\varrho_\epsilon)\varrho_\epsilon\mathbf{v}_\epsilon \cdot \nabla\varrho_\epsilon \frac{1}{\varrho_\epsilon + \delta}\,\mathrm{d}x \ge \int_\Omega \big(\epsilon m K(\varrho_\epsilon) - \epsilon\varrho_\epsilon\big)\big(\log k - \log(\varrho_\epsilon + \delta)\big)\,\mathrm{d}x.$$

Passing to the limit with $\delta \to 0^+$ in the last equation we get

$$\int_\Omega K(\varrho_\epsilon)\mathbf{v}_\epsilon \cdot \nabla\varrho_\epsilon\,\mathrm{d}x \ge \epsilon \int_\Omega \big(mK(\varrho_\epsilon) - \varrho_\epsilon\big)(\log k - \log\varrho_\epsilon)\,\mathrm{d}x,$$

hence

$$-\int_\Omega \varrho_\epsilon \operatorname{div} \mathbf{v}_\epsilon \, \mathrm{d}x \geq \int_\Omega \big(1 - K(\varrho_\epsilon)\big)\mathbf{v}_\epsilon \cdot \nabla \varrho_\epsilon \, \mathrm{d}x + \epsilon \int_\Omega mK(\varrho_\epsilon) \log \frac{k}{\varrho_\epsilon} \, \mathrm{d}x.$$

Thus, by (2.57)

$$\int_\Omega \varrho_\epsilon \operatorname{div} \mathbf{v}_\epsilon \, \mathrm{d}x \leq R_\epsilon,$$

with $R_\epsilon \to 0$ as $\epsilon \to 0+$. Hence the definition of G yields

$$\int_\Omega p_\epsilon(\varrho_\epsilon, c_\epsilon)\varrho_\epsilon \, \mathrm{d}x \leq \int_\Omega G_\epsilon \varrho_\epsilon \, \mathrm{d}x - (2\mu + \lambda) R_\epsilon$$

and passing to the limit with ϵ we get (2.63), since according to Lemma 2.14 we have $\overline{G\varrho} = G\varrho$. □

Lemma 2.17.

$$\int_\Omega \overline{p(\varrho, c)}\varrho \, \mathrm{d}x = \int_\Omega G\varrho \, \mathrm{d}x. \tag{2.64}$$

Proof. First, with the use of Fridrichs' commutator lemma 4.23 we are able to approximate ϱ by smooth bounded functions ϱ_n such that $\varrho_n \to \varrho$ in $L^p(\Omega)$, for any $p < \infty$ and

$$\int_\Omega (\varrho_n \operatorname{div} \mathbf{v} + \mathbf{v} \cdot \nabla \varrho_n) \, \mathrm{d}x = 0,$$

see [100, Lemma 4.4]. Further testing the limit continuity equation by a smooth function $\log(\varrho_n + \delta) - \log \delta$ yields

$$\int_\Omega \frac{1}{\varrho_n + \delta} \varrho \mathbf{v} \cdot \nabla \varrho_n \, \mathrm{d}x = 0,$$

we pass to the limit at first with $n \to \infty$

$$\int_\Omega \frac{\varrho}{\varrho + \delta} \mathbf{v} \cdot \nabla \varrho \, \mathrm{d}x = 0,$$

and then with $\delta \to 0^+$ hence we obtain $\int_\Omega \mathbf{v} \cdot \nabla \varrho \, \mathrm{d}x = 0$, i.e. $\int_\Omega \varrho \operatorname{div} \mathbf{v} \, \mathrm{d}x = 0$ as well. Thus, by multiplying the definition of G by ϱ and integrating the resulting relation over Ω we conclude (2.64). □

Further, the strong convergence of c and the convexity of mappings $s \mapsto s^\gamma$ and $s \mapsto s^2 l_\epsilon'(s)$ gives us by Theorem 4.18

$$\overline{p(\varrho, c)}\,\varrho \leq \overline{p(\varrho, c)\varrho},$$

which combined with (2.63) and (2.64) yields

$$\overline{p(\varrho, c)\varrho} = \overline{p(\varrho, c)}\,\varrho \;\text{ a.e. in } \Omega.$$

Thus, $\overline{\varrho^{\gamma+1}} = \varrho\overline{\varrho^\gamma}$ and monotonicity argument from Lemma 4.17 yields

$$\varrho_\epsilon \to \varrho \text{ strongly in } L^\gamma(\Omega).$$

Finally, we move our attention to the last two equations of (2.1)–(2.4) and show that due to strong convergence of ϱ_ϵ and c_ϵ, all the remaining nonlinearities can be identified, so we have indeed obtained the solutions to our original system. This completes the proof of Theorem 2.2.

2.5 Existence for $\gamma > 3$

In order to prove the existence of only weak solutions of the problem for $\gamma \in (3,6]$, as presented in Theorem 2.3, we will use the following idea. We modify the pressure

$$p_\delta(\varrho, c) = p(\varrho, c) + \delta \varrho^\Gamma$$

with $\Gamma > 6$, for which the already proven result stays obviously valid and then using the a priori estimates derived in Section 2.2 pass to the limit with $\delta \to 0^+$. This limit passage will be performed in the same spirit as in the case of the Navier–Stokes system, using the techniques due to Lions and Feireisl, see e.g. [115] with θ replaced by c. The compactness of the additional stress in the momentum equation and in the additional equations will be just easy application of the Rellich–Kondrachov compactness theorem due to the uniform bound of Δc.

First, according to the already proven part there exists a sequence of solutions satisfying the equations with the modified pressure p_δ denoted by $(\varrho_\delta, \mathbf{v}_\delta, c_\delta, \overset{+}{c}_\delta)$. Exactly with the same procedure as in Lemma 2.5 we can deduce that it satisfies

$$\begin{aligned} &\|\mathbf{v}_\delta\|_{W^{1,2}(\Omega)} + \|\varrho_\delta\|_{L^{3\gamma-6}(\Omega)} + \delta \|\varrho_\delta\|^{\Gamma+2\gamma-6}_{L^{\Gamma+2\gamma-6}(\Omega)} + \|\nabla c_\delta\|_{L^{\frac{6\gamma-12}{\gamma}}(\Omega)} \\ &\qquad + \left\|\nabla^2 c_\delta\right\|_{L^{\frac{6\gamma-12}{3\gamma-4}}(\Omega)} + \left\|\overset{+}{c}_\delta\right\|_2 + \|\varrho_\delta L'(c_\delta)\|_{L^{\frac{6\gamma-12}{3\gamma-4}}(\Omega)} \leq C, \end{aligned} \tag{2.65}$$

with C independent of δ. So we can extract subsequences, denoted here in the same way, such that

$$\begin{aligned} \mathbf{v}_\delta &\rightharpoonup \mathbf{v} \text{ in } W^{1,2}(\Omega), & \mathbf{v}_\delta &\to \mathbf{v} \text{ in } L^q(\Omega), \text{ for all } 1 < q < 6 \\ \varrho_\delta &\rightharpoonup \varrho \text{ in } L^{3\gamma-6}(\Omega), & \varrho_\delta^\gamma &\rightharpoonup \overline{\varrho^\gamma} \text{ in } L^r(\Omega), \text{ for some } r > 1 \\ c_\delta &\rightharpoonup c \text{ in } W^{2,\frac{6\gamma-12}{3\gamma-4}}(\Omega), & \nabla c_\delta &\to \nabla c \text{ in } L^2(\Omega), \\ \varrho_\delta L'(c_\delta) &\rightharpoonup \varrho L'(c) \text{ in } L^{6/5}(\Omega), & \delta \varrho_\delta &\to 0 \text{ in } L^r(\Omega), \text{ for all } 1 < r < \Gamma + 3\gamma - 6 \\ \overset{+}{c}_\delta &\rightharpoonup \overset{+}{c} \text{ in } L^2(\Omega), & \varrho_\delta \overset{+}{c}_\delta &\rightharpoonup \overline{\varrho \overset{+}{c}} \text{ in } L^2(\Omega). \end{aligned}$$

Hence consequently,

$$\begin{aligned} \varrho_\delta \mathbf{v}_\delta &\rightharpoonup \varrho \mathbf{v} \text{ in } L^2(\Omega), & \varrho_\delta \mathbf{v}_\delta \otimes \mathbf{v}_\delta &\rightharpoonup \varrho \mathbf{v} \otimes \mathbf{v} \text{ in } L^{3/2}(\Omega), \\ \varrho_\delta \log \varrho_\delta &\rightharpoonup \overline{\varrho \log \varrho} \text{ in } L^3(\Omega), & \varrho_\delta c_\delta &\rightharpoonup \varrho c \text{ in } L^3(\Omega), \\ \nabla c_\delta \otimes \nabla c_\delta &\to \nabla c \otimes \nabla c \text{ in } L^1(\Omega), & |\nabla c_\delta|^2 &\to |\nabla c|^2 \text{ in } L^1(\Omega). \end{aligned}$$

Note that c_δ are continuous and $c_\delta \in [0,1]$ a.e. in Ω, hence $\|c_\delta\|_{L^\infty} \leq 1$ independently of $\delta > 0$. To summarize, we have shown that the limit solution satisfies in the weak sense in particular

$$\begin{gathered} \operatorname{div}(\varrho \mathbf{v}) = 0, \\ \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbb{S}(\nabla \mathbf{v}) + \nabla \overline{p(\varrho, c)} = \varrho \mathbf{g} + \operatorname{div}\Big(\nabla c \otimes \nabla c - \frac{|\nabla c|^2}{2}\mathbb{I}\Big) \text{ in } \Omega, \\ \mathbf{v} \cdot \mathbf{n} = 0, \\ \mathbf{n} \cdot \mathbb{T}(c, \mathbf{v}) \cdot \boldsymbol{\tau}^n + \alpha \mathbf{v} \cdot \boldsymbol{\tau}^n = 0 \text{ on } \partial\Omega. \end{gathered}$$

Thus, the main difficulty is to show that

$$\varrho_\delta \to \varrho \text{ in } L^s(\Omega) \text{ for some } s \geq 1.$$

To deal with this problem, we can proceed exactly in the same manner as in the standard case of the Navier–Stokes equations. Moreover, by virtue of the constraint $\gamma > 3$, the limit renormalized continuity equation is satisfied according to Fridrichs' commutator lemma 4.23.

In order to prove the celebrated *effective viscous flux identity*

$$\overline{p(\varrho,c)\,T_k(\varrho)} - (2\mu+\lambda)\overline{\operatorname{div}\mathbf{v}\,T_k(\varrho)} = \overline{p(\varrho,c)}\;\overline{T_k(\varrho)} - (2\mu+\lambda)\operatorname{div}\mathbf{v}\,\overline{T_k(\varrho)}, \tag{2.66}$$

where

$$T_k(z) = kT\Big(\frac{z}{k}\Big), \qquad T(z) = \begin{cases} z \text{ for } 0 \leq z \leq 1, \\ \text{concave on } (0,\infty), \\ 2 \text{ for } z \geq 3, \end{cases}$$

we will test for $k \in \mathbb{N}$, and $\zeta \in C_c^\infty(\Omega)$ the momentum equation and its approximate version by[12]

$$\boldsymbol{\varphi} = \zeta\nabla\Delta^{-1}\big(\chi_\Omega\overline{T_k(\varrho)}\big), \qquad \text{and} \qquad \boldsymbol{\varphi}_\delta = \zeta\nabla\Delta^{-1}\big(\chi_\Omega T_k(\varrho_\delta)\big),$$

respectively. Taking difference of the resulting equalities yields

$$\begin{aligned} \lim_{\delta\to0+}\int_\Omega &\zeta\Big(p_\delta(\varrho_\delta,c_\delta)T_k(\varrho_\delta) - \mathbb{S}(\nabla\mathbf{v}_\delta) : \mathcal{R}[T_k(\varrho_\delta)\chi_\Omega]\Big)\,\mathrm{d}x \\ &= \int_\Omega \zeta\,\Big(\overline{p(\varrho)}\;\overline{T_k(\varrho)} - \mathbb{S}(\nabla\mathbf{v}) : \mathcal{R}[\overline{T_k(\varrho)}\chi_\Omega]\Big)\,\mathrm{d}x \\ &\quad + \lim_{\delta\to0+}\int_\Omega \zeta\Big(T_k(\varrho_\delta)\mathbf{v}_\delta\cdot\mathcal{R}[\varrho_\delta\mathbf{v}_\delta\chi_\Omega] - \varrho_\delta(\mathbf{v}_\delta\otimes\mathbf{v}_\delta) : \mathcal{R}[T_k(\varrho_\delta)\chi_\Omega]\Big)\,\mathrm{d}x \\ &\qquad - \int_\Omega \zeta\Big(\overline{T_k(\varrho)}\mathbf{v}\cdot\mathcal{R}[\varrho\mathbf{v}\chi_\Omega] - \varrho(\mathbf{v}\otimes\mathbf{v}) : \mathcal{R}[\overline{T_k(\varrho)}\chi_\Omega]\Big)\,\mathrm{d}x, \end{aligned} \tag{2.67}$$

where we have already subtracted the terms convergent due to basic properties of the pseudodifferential operators. From Lemma 4.21 we further deduce

$$\begin{aligned} \int_\Omega \zeta\mathbf{v}_\delta\cdot\Big(T_k(\varrho_\delta)\mathcal{R}[\varrho_\delta\mathbf{v}_\delta\chi_\Omega] - \varrho_\delta\mathcal{R}[T_k(\varrho_\delta)\chi_\Omega]\mathbf{v}_\delta\Big)\,\mathrm{d}x& \\ \to \int_\Omega \zeta\mathbf{v}\cdot\Big(\overline{T_k(\varrho)}\mathcal{R}[\varrho\mathbf{v}\chi_\Omega] - \varrho\mathcal{R}[\overline{T_k(\varrho)}\chi_\Omega]\mathbf{v}\Big)\,\mathrm{d}x&, \end{aligned} \tag{2.68}$$

thus (2.67) reduces to

$$\begin{aligned} \int_\Omega \zeta\,\Big(\overline{p(\varrho,c)T_k(\varrho)} - \overline{p(\varrho,c)}\;\overline{T_k(\varrho)}\Big)\,\mathrm{d}x& \\ = \int_\Omega \Big(\zeta(2\mu+\lambda)\overline{\operatorname{div}\mathbf{v}T_k(\varrho)} - \zeta(2\mu+\lambda)\overline{T_k(\varrho)}\operatorname{div}\mathbf{v}\Big)\,\mathrm{d}x&, \end{aligned} \tag{2.69}$$

[12]Let us recall the notation χ_Ω for a characteristic function of Ω, and $\nabla\Delta^{-1}$ for the inverse divergence defined through (4.107).

for more detailed proof see [110, Lemma 6.3]. Further, to exploit the renormalized continuity equation we set

$$b_k(t) = \begin{cases} t\log t & \text{for } t \in [0,k], \\ t\log k + t - k & \text{for } t > k, \end{cases}$$

hence $tb_k'(t) - b_k(t) = T_k(t)$ and using Lemma 4.24

$$\int_\Omega \overline{T_k(\varrho)\operatorname{div}\mathbf{v}}\,\mathrm{d}x = 0, \qquad \int_\Omega \overline{T_k(\varrho)}\operatorname{div}\mathbf{v}\,\mathrm{d}x = 0. \tag{2.70}$$

Further, from simple algebraic inequality $(t-s)^\gamma \le t^\gamma - s^\gamma$, $t \ge s \ge 0$, we deduce that

$$\begin{aligned}
\limsup_{\delta\to 0+} \int_\Omega & |T_k(\varrho_\delta) - T_k(\varrho)|^{\gamma+1}\,\mathrm{d}x \\
&\le \limsup_{\delta\to 0+} \int_\Omega (\varrho^\gamma - \varrho_\delta^\gamma)\big(T_k(\varrho) - T_k(\varrho_\delta)\big)\,\mathrm{d}x \\
&= \int_\Omega \overline{\varrho^\gamma T_k(\varrho)} - \overline{\varrho^\gamma}\,\overline{T_k(\varrho)}\,\mathrm{d}x + \int_\Omega \big(\varrho^\gamma - \overline{\varrho^\gamma}\big)\big(T_k(\varrho) - \overline{T_k(\varrho)}\big)\,\mathrm{d}x,
\end{aligned}$$

where the second integral on the right-hand side is non-positive, since $t \mapsto t^\gamma$ is convex and $t \mapsto T_k(t)$ is concave. Thus,

$$\begin{aligned}
&\limsup_{\delta\to 0+} \int_\Omega |T_k(\varrho_\delta) - T_k(\varrho)|^{\gamma+1}\,\mathrm{d}x \\
&\le \int_\Omega \overline{p(\varrho,c)T_k(\varrho)} - \overline{p(\varrho,c)}\,\overline{T_k(\varrho)}\,\mathrm{d}x + \int_\Omega \overline{\varrho(ac+d)}\,\overline{T_k(\varrho)} - \overline{\varrho(ac+d)T_k(\varrho)}\,\mathrm{d}x.
\end{aligned} \tag{2.71}$$

However, since c converges strongly, and since we have due to Theorem 4.17 $\overline{\varrho T_k(\varrho)} - \varrho\overline{T_k(\varrho)} \ge 0$, the second integral in (2.71) is non-positive. Therefore, according to the effective viscous flux identity (2.66) and (2.70)

$$\begin{aligned}
\limsup_{\delta\to 0+} \int_\Omega |T_k(\varrho_\delta) - T_k(\varrho)|^{\gamma+1}\,\mathrm{d}x &\le (2\mu+\lambda)\int_\Omega \Big(\overline{T_k(\varrho)\operatorname{div}\mathbf{v}} - \overline{T_k(\varrho)}\operatorname{div}\mathbf{v}\Big)\,\mathrm{d}x \\
&\le C\int_\Omega \big|T_k(\varrho) - \overline{T_k(\varrho)}\big|\,|\operatorname{div}\mathbf{v}|\,\mathrm{d}x \\
&\le C\big\|\overline{T_k(\varrho)} - T_k(\varrho)\big\|_{L^2(\Omega)}\|\operatorname{div}\mathbf{v}\|_{L^2(\Omega)},
\end{aligned}$$

so by (2.65) the left-hand side is uniformly bounded with respect to k and we can interpolate

$$\big\|\overline{T_k(\varrho)} - T_k(\varrho)\big\|_{L^2(\Omega)} \le C\big\|\overline{T_k(\varrho)} - T_k(\varrho)\big\|_{L^1(\Omega)}^{\frac{\gamma-1}{2\gamma}}\big\|\overline{T_k(\varrho)} - T_k(\varrho)\big\|_{L^{\gamma+1}(\Omega)}^{\frac{\gamma+1}{2\gamma}}$$

in order to obtain

$$\limsup_{\delta\to 0+} \int_\Omega |T_k(\varrho_\delta) - T_k(\varrho)|^{\gamma+1}\,\mathrm{d}x \le C\big\|\overline{T_k(\varrho)} - T_k(\varrho)\big\|_{L^1(\Omega)}^{\frac{\gamma-1}{2\gamma-1}}.$$

Furthermore, $\left\|\overline{T_k(\varrho)} - T_k(\varrho)\right\|_{L^1(\Omega)} \le \left\|\overline{T_k(\varrho)} - \varrho\right\|_{L^1(\Omega)} + \left\|\varrho - T_k(\varrho)\right\|_{L^1(\Omega)}$, so

$$\lim_{k\to\infty} \left\|\overline{T_k(\varrho)} - T_k(\varrho)\right\|_{L^1(\Omega)} = 0,$$

and consequently also

$$\lim_{k\to\infty} \limsup_{\delta\to 0^+} \|T_k(\varrho_\delta) - T_k(\varrho)\|_{L^{\gamma+1}(\Omega)} = 0.$$

Finally,

$$\begin{aligned}\limsup_{\delta\to 0^+}\left\|\varrho_\delta - \varrho\right\|_{L^1(\Omega)} \le \limsup_{\delta\to 0^+}\left\|\varrho_\delta - T_k(\varrho_\delta)\right\|_{L^1(\Omega)} + \limsup_{\delta\to 0^+}\left\|T_k(\varrho_\delta) - T_k(\varrho)\right\|_{L^1(\Omega)} \\ + \limsup_{\delta\to 0^+}\left\|T_k(\varrho) - \varrho\right\|_{L^1(\Omega)} \;=\; 0, \quad (2.72)\end{aligned}$$

which implies strong convergence of densities.

Lastly, we turn our attention back to the Allen–Cahn equation and realize that as soon as we have pointwise convergence of c and ϱ and weak convergence of $\overset{+}{c}$, the fact that the limit of the sequence satisfies the original equations is immediate. This completes the proof of Theorem 2.3.

3. Steady strong solutions to the Navier–Stokes system with density-dependent viscosity

In this chapter, we study the steady version of the Navier–Stokes system for compressible fluid, neglecting the thermal effects, id est we take in our general model $c \equiv const.$ and $\vartheta \equiv const.$ and consider

$$\operatorname{div}(\varrho \mathbf{v}) = 0, \tag{3.1}$$
$$\operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v}) = \operatorname{div} \mathbb{T} + \varrho \mathbf{g}. \tag{3.2}$$

Correspondingly, the stress tensor $\mathbb{T}$ can be written as

$$\mathbb{T} = 2\mu(\varrho)\mathbb{D}(\mathbf{v}) + \lambda(\varrho) \operatorname{div} \mathbf{v}\mathbb{I} - p(\varrho)\mathbb{I}. \tag{3.3}$$

We suppose the pressure to be of the form $p(\varrho) = \varrho^\gamma$ with the viscosity coefficients

$$\mu(\varrho) = \varrho, \ \lambda(\varrho) = 0. \tag{3.4}$$

Note that for $d = 2$ and $\gamma = 2$ we obtain a system which is formally equivalent to the so-called shallow water equations, see e.g. [89]. We consider the system with the slip boundary condition for the velocity

$$\mathbf{v} \cdot \mathbf{n} = 0 \text{ at } \partial\Omega, \tag{3.5}$$
$$\mathbf{n} \cdot \mathbb{T}(\varrho, \nabla \mathbf{v}) \cdot \boldsymbol{\tau}^k + \alpha \mathbf{v} \cdot \boldsymbol{\tau}^k = 0 \text{ at } \partial\Omega, \tag{3.6}$$

where $\boldsymbol{\tau}^k$, $k = 1, 2$ are two linearly independent tangent vectors to $\partial\Omega$, $\mathbf{n}$ denotes the normal vector and $\alpha \geq 0$ represents the friction on the boundary.

3.1 The main result

We are interested in the existence of solutions with large density, hence we assume $\varrho = m + r$, where $\int_\Omega r \; dx = 0$ and $\frac{1}{|\Omega|} \int_\Omega \varrho \; dx = m$, with m large enough. More precisely, we define for $p > d = 3$

$$\Xi = m^{\gamma-2} \|r\|_{W^{1,p}(\Omega)} + \|\mathbf{v}\|_{W^{2,p}(\Omega)} \tag{3.7}$$

and consider solutions for which

$$m \gg \Xi + \|\mathbf{g}\|_{L^p(\Omega)} . \tag{3.8}$$

More precise version of condition (3.8) will be given later. Our result is inspired by the regularity result of Lions, see [85, Theorem 6.17] and the successive discussion therein. It guarantees that under the condition

$$\operatorname{ess\,inf} \varrho > 0$$

one can expect higher regularity of the solutions. It is worth mentioning here a similar existence result for small Mach number due to Choe and Jin [19], see also [27] for the heat-conducting case. They needed to assume for the existence quite smooth (H^2) external force, and also constant viscosity coefficients. Here, we work in the L^p framework for the higher order estimates, which allows us to assume much less about the external force, in fact L^p is enough. Further, we take density-dependent viscosity coefficients, and slip boundary condition for velocity, both of these assumptions seems to be essential for our approach. It is based on simple algebraic relation between the continuity equation and the potential part of the momentum equation, mimicking the method of decomposition due to Novotný and Padula [113].

Our main result reads as follows.

Theorem 3.1. *Let $\gamma > 1$, $\alpha \geq 0$ and let μ and λ satisfy (3.4). Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain which is not axially symmetric, $\mathbf{g} \in L^p(\Omega)$ for some $p \in (3, 6)$. Suppose further that m is sufficiently large with respect to certain norms of $\mathbf{g}$ in the sense of condition (3.25). Then there exists at least one strong solution to the Navier–Stokes equations (3.1)–(3.3) with boundary conditions (3.5)–(3.6) in the class $(\varrho, \mathbf{v}) \in W^{1,p}(\Omega) \times W^{2,p}(\Omega)$.*

Remark 3.2. Our result remains true in 2D as well, we give however the proof only in the more complicated 3D case. Similarly, we could take the second viscosity coefficient λ positive as well, but it would bring only unnecessary technicalities. Finally, in the case of axially symmetric domain, in order to get the basic energy estimate independent of m, it would be necessary to consider $\alpha > 0$ of the same order as m, e.g. as a linear function of density ϱ.

First, we will deduce the a priori estimates.

3.2 A priori bounds

The system can be rewritten as

$$m \operatorname{div} \mathbf{v} + \mathbf{v} \cdot \nabla r + r \operatorname{div} \mathbf{v} = 0, \tag{3.9}$$

$$(m + r)\, \mathbf{v} \cdot \nabla \mathbf{v} - 2 \operatorname{div}\big((m + r)\mathbb{D}(\mathbf{v})\big) + \gamma(m + r)^{\gamma-1}\nabla r = \varrho \mathbf{g} \text{ in } \Omega, \tag{3.10}$$

$$\mathbf{v} \cdot \mathbf{n} = 0, \qquad \mathbf{n} \cdot (m + r)\mathbb{D}(\mathbf{v}) \cdot \boldsymbol{\tau}^k + \alpha \mathbf{v} \cdot \boldsymbol{\tau}^k = 0 \text{ on } \partial\Omega. \tag{3.11}$$

The basic energy estimate reads

$$\int_\Omega m \,|\mathbb{D}(\mathbf{v})|^2 \,\mathrm{d}x + \sum_{k=1}^{2} \int_{\partial\Omega} \alpha \left|\mathbf{v} \cdot \boldsymbol{\tau}^k\right|^2 \mathrm{d}S \leq C \int_\Omega \left(|\varrho \mathbf{g} \cdot \mathbf{v}| + |r|\, |\mathbb{D}(\mathbf{v})|^2\right) \mathrm{d}x,$$

with C independent of m. Thus, assuming $m \gg \Xi$, we obtain by the Korn inequality from Theorem 4.4

$$\|\mathbf{v}\|_{W^{1,2}(\Omega)} \leq C \,\|\mathbf{g}\|_{L^{6/5}(\Omega)} . \tag{3.12}$$

Further, we test the momentum equation with a function $-\boldsymbol{\Phi}$, $\boldsymbol{\Phi} = \mathcal{B}[r]$, where $\mathcal{B}$ denotes the Bogovskii operator from Theorem 4.13, accordingly we have for $\boldsymbol{\Phi}$ the estimate $\|\boldsymbol{\Phi}\|_6 \leq C \|\nabla \boldsymbol{\Phi}\|_2 \leq C \|r\|_2$. This yields

$$(\gamma m^{\gamma-1} - \|r\|_\infty) \,\|r\|_2^2 \leq Cm\Big(\|\nabla \mathbf{v}\|_2 \,\|\nabla \boldsymbol{\Phi}\|_2 + \|\mathbf{v}\|_3 \,\|\nabla \mathbf{v}\|_2 \,\|\boldsymbol{\Phi}\|_6 + \|\mathbf{g}\|_{6/5} \,\|\boldsymbol{\Phi}\|_6 \Big), \tag{3.13}$$

hence using Young's inequality and (3.12)

$$m^{\gamma-2}\|r\|_2^2 \le \frac{C}{m^{\gamma-2}}\Big(\|\mathbf{g}\|_{6/5}^4 + \|\mathbf{g}\|_{6/5}^2\Big).$$

To proceed, in order to recover the effective viscous flux, we will use the Helmholtz decomposition from (4.106), in order to estimate the solenoidal and the gradient part of the momentum equation separately. We will denote for the sake of clarity

$$\mathcal{G} = -\varrho\mathbf{v}\cdot\nabla\mathbf{v} + 2\operatorname{div}(r\mathbb{D}(\mathbf{v})) + \varrho\mathbf{g}.$$

First, applying curl on (3.10) yields for $\boldsymbol{\omega} = \operatorname{curl}\mathbf{v}$, see [132]

$$\begin{aligned}
-m\Delta\boldsymbol{\omega} &= \operatorname{curl}\mathcal{G} \text{ in } \Omega,\\
\operatorname{div}\boldsymbol{\omega} &= 0,\\
\boldsymbol{\omega}\cdot\boldsymbol{\tau}^1 &= -\Big(2\chi_2 - \frac{\alpha}{m+r}\Big)\mathbf{v}\cdot\boldsymbol{\tau}^2,\\
\boldsymbol{\omega}\cdot\boldsymbol{\tau}^2 &= \Big(2\chi_1 - \frac{\alpha}{m+r}\Big)\mathbf{v}\cdot\boldsymbol{\tau}^1 \text{ on } \partial\Omega,
\end{aligned}$$

with χ_i denoting the curvatures corresponding to the directions $\boldsymbol{\tau}^i$. For more details concerning the relations between formulations of slip boundary conditions see [96]. Thus, according to the elliptic regularity theory, see Theorem 4.11

$$\|\boldsymbol{\omega}\|_{W^{1,p}(\Omega)} \le C\left(\frac{\|\operatorname{curl}\mathcal{G}\|_{(W^{1,p'}(\Omega))^*}}{m} + \|\mathbf{v}\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} + \left\|\alpha\mathbf{v}(m+r)^{-1}\right\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}\right). \tag{3.14}$$

The boundary term can be estimated as follows

$$\begin{aligned}
&\|\mathbf{v}\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} + \left\|\alpha\mathbf{v}(m+r)^{-1}\right\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}\\
&\qquad\le C(\|\mathbf{v}\|_{W^{1,p}(\Omega)} + \alpha m^{-2}\|\mathbf{v}\|_{L^\infty(\Omega)}\|r\|_{W^{1,p}(\Omega)}).
\end{aligned} \tag{3.15}$$

Further, we decompose the velocity field $\mathbf{v} = P_H\mathbf{v} + \nabla P_\nabla\mathbf{v}$ using the Helmholtz decomposition from Theorem 4.19. The solenoidal part of the velocity field $P_H\mathbf{v}$ satisfies the overdetermined system

$$\begin{aligned}
\operatorname{curl} P_H\mathbf{v} &= \boldsymbol{\omega} \text{ in } \Omega,\\
\operatorname{div} P_H\mathbf{v} &= 0 \text{ in } \Omega,\\
P_H\mathbf{v}\cdot\mathbf{n} &= 0 \text{ on } \partial\Omega,
\end{aligned} \tag{3.16}$$

so we obtain due to Theorem 4.14

$$\left\|\nabla^2 P_H\mathbf{v}\right\|_{L^p(\Omega)} \le C\left(\frac{\|\mathcal{G}\|_{L^p(\Omega)}}{m} + \|\nabla\mathbf{v}\|_{L^p(\Omega)} + \frac{1}{m^2}\|\mathbf{v}\|_{L^\infty(\Omega)}\|r\|_{W^{1,p}(\Omega)}\right). \tag{3.17}$$

Similarly, the potential part of the momentum equation (3.10) reads[1]

$$p(\varrho) - \{p(\varrho)\}_\Omega - 2m\operatorname{div}\mathbf{v} = P_\nabla\Big(\mathcal{G} + m\Delta P_H\mathbf{v}\Big). \tag{3.18}$$

[1] As above, we denote $\{g\}_\Omega = \frac{1}{|\Omega|}\int_\Omega g\,\mathrm{d}x$

We can use the Taylor expansion, in order to observe

$$p(\varrho) = (m+r)^\gamma = m^\gamma + \gamma m^{\gamma-1} r + \frac{1}{2} p''(\zeta) r^2, \tag{3.19}$$

where ζ lies between m and $m+r$, whence $|p''(\zeta)r^2| \le C m^{\gamma-2} r^2$. Subtracting the average from (3.19) yields

$$p(\varrho) - \{p(\varrho)\}_\Omega = \gamma m^{\gamma-1} r + \frac{1}{2}\big(p''(\zeta)r^2 - \{p''(\zeta)r^2\}_\Omega\big). \tag{3.20}$$

We can combine

$$\gamma m^{\gamma-1} r - 2m \operatorname{div} \mathbf{v} + \frac{1}{2}\big(p''(\zeta)r^2 - \{p''(\zeta)r^2\}_\Omega\big) = P_\triangledown\Big(\mathcal{G} + m\Delta P_H \mathbf{v}\Big) \tag{3.21}$$

with the continuity equation

$$m \operatorname{div} \mathbf{v} + \mathbf{v}\cdot\nabla r + r \operatorname{div} \mathbf{v} = 0,$$

in order to get

$$\gamma m^{\gamma-1} r + 2\nabla r\cdot \mathbf{v} = -2r \operatorname{div} \mathbf{v} + P_\triangledown\Big(\mathcal{G} + m\Delta P_H \mathbf{v}\Big) - \frac{1}{2}\big(p''(\zeta)r^2 - \{p''(\zeta)r^2\}_\Omega\big),$$

and after differentiating

$$\begin{aligned} \gamma m^{\gamma-1}\nabla r + 2\mathbf{v}\cdot\nabla\nabla r &= -2\nabla r \operatorname{div} \mathbf{v} - 2r\nabla \operatorname{div} \mathbf{v} - 2\nabla\mathbf{v}\nabla r \\ &\quad - \frac{1}{2}\nabla\big(p''(\zeta)r^2\big) + \nabla P_\triangledown\Big(\mathcal{G} + m\Delta P_H \mathbf{v}\Big). \end{aligned} \tag{3.22}$$

Note that $P_\triangledown$ is continuous from L^p to $W^{1,p}$, so $\nabla P_\triangledown$ is actually zero order operator. In order to obtain from (3.22) the required information about ∇r, we test the k-th component of (3.22) by $\partial_k r\, |\partial_k r|^{p-2}$. The second term on the left-hand side can be then rewritten using integration by parts as, see [111, Lemma 2.3]

$$\int_\Omega \mathbf{v}\cdot\nabla\partial_k r\, |\partial_k r|^{p-2}\, \partial_k r \,\mathrm{d}x = -\frac{1}{p}\int_\Omega \operatorname{div} \mathbf{v}\, |\partial_k r|^p \,\mathrm{d}x;$$

$\left|\nabla\big(p''(\zeta)r^2\big)\right| \le C m^{\gamma-2}\,|r|\,|\nabla r|$. Thus, with usage of (3.17) and the fact that $\|\nabla \mathbf{v}\|_{L^\infty(\Omega)} \ll m^{\gamma-1}$,

$$m^{\gamma-1}\,\|\nabla r\|_{L^p(\Omega)} \le C\Big(\|\mathcal{G}\|_{L^p(\Omega)} + m\,\big\|\nabla^2 P_H \mathbf{v}\big\|_{L^p(\Omega)}\Big). \tag{3.23}$$

Moreover, using (3.21), we can bound the potential part of the velocity. Since

$$2m\nabla \operatorname{div} \mathbf{v} = \gamma m^{\gamma-1}\nabla r + \frac{1}{2}\nabla\big(p''(\zeta)r^2\big) - \nabla P_\triangledown\Big(\mathcal{G} + m\Delta P_H \mathbf{v}\Big),$$

we obtain for the quantity $\nabla \operatorname{div} \mathbf{v}$ similar estimate, namely

$$m\big\|\nabla \operatorname{div} \mathbf{v}\big\|_p \le C\big(m^{\gamma-1}\big\|\nabla r\big\|_p + \big\|\mathcal{G}\big\|_p + m\big\|\nabla^2 P_H \mathbf{v}\big\|_p\big). \tag{3.24}$$

Putting together (3.17), (3.23) and (3.24) yields

$$\Xi \le \frac{C}{m}\Big(\|\mathcal{G}\|_{L^p(\Omega)} + m\,\|\nabla \mathbf{v}\|_{L^p(\Omega)} + \frac{1}{m}\,\|\mathbf{v}\|_{L^\infty(\Omega)}\,\|r\|_{W^{1,p}(\Omega)}\Big).$$

The second term can be eliminated by means of the Gagliardo–Nirenberg interpolation inequality

$$\|\nabla \mathbf{v}\|_{L^p(\Omega)} \leq C \|\nabla \mathbf{v}\|_{L^2(\Omega)}^{\frac{\frac{3p-6}{5p-6}}{}} \left\|\nabla^2 \mathbf{v}\right\|_{L^p(\Omega)}^{\frac{2p}{5p-6}},$$

while the last one can be for $\gamma > 1$ put directly to the left-hand side for sufficiently large m. Recalling the definition of $\mathcal{G}$, one can see that the most restrictive term is, except the external force, the convective term. We will estimate it for $p \in (3,6)$ by means of interpolation and energy inequality (3.12) as follows

$$\begin{aligned}\left\|\varrho \mathbf{v} \cdot \nabla \mathbf{v}\right\|_p \leq \left\|\varrho\right\|_\infty \left\|\mathbf{v}\right\|_6 \left\|\nabla \mathbf{v}\right\|_{\frac{6p}{6-p}} &\leq \left(m + \left\|r\right\|_\infty\right) \left\|\mathbf{v}\right\|_6 \left\|\nabla \mathbf{v}\right\|_2^{\frac{6-p}{3p}} \left\|\nabla \mathbf{v}\right\|_\infty^{\frac{4p-6}{3p}} \\ &\leq Cm \left\|\mathbf{g}\right\|_{6/5}^{\frac{2p+6}{3p}} \left\|\nabla^2 \mathbf{v}\right\|_p^{\frac{4p-6}{3p}}.\end{aligned}$$

To sum up, we get for $p < 6$

$$\Xi \leq C\left(\left\|\mathbf{g}\right\|_p + \left\|\mathbf{g}\right\|_{6/5}^{\frac{2p+6}{6-p}}\right) = C_{\mathbf{g}}. \tag{3.25}$$

Thus, under the assumption $\gamma > 1$, we obtain the a priori estimate

$$\left\|\nabla^2 \mathbf{v}\right\|_{W^{2,p}(\Omega)} + m^{\gamma-2} \left\|\nabla r\right\|_{L^p(\Omega)} = \Xi \leq C_{\mathbf{g}}. \tag{3.26}$$

The basic idea is to take m sufficiently larger than the right-hand side of (3.26), id est $C_{\mathbf{g}} \ll m$. Finally, we can look back on the continuity equation (3.9), and conclude from (3.26) that

$$\|\operatorname{div} \mathbf{v}\|_{L^p(\Omega)} \leq 2\frac{C_{\mathbf{g}}^2}{m^{\gamma-1}},$$

which expresses in a quantitative way the idea that we are in fact close to the incompressible case.

3.3 Approximation

Let us denote the classes in which we will search for the solution

$$\begin{aligned} M_r(m) =& \left\{ f \in W^{1,p}(\Omega), \int_\Omega f \,\mathrm{d}x = 0, \left\|f\right\|_\infty + \left\|\nabla f\right\|_p \leq C_{\mathbf{g}} m^{2-\gamma} \right\}, \\ M_{\mathbf{v}}(m) =& \Big\{ \mathbf{f} \in W^{2,p}(\Omega, \mathbb{R}^3),\ \mathbf{f} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega, \\ & \left\|\nabla \mathbf{f}\right\|_2 \leq E,\ \left\|\nabla \mathbf{f}\right\|_\infty + \left\|\mathbf{f}\right\|_\infty + \left\|\nabla^2 \mathbf{f}\right\|_p \leq C_{\mathbf{g}},\ m^{\gamma-1} \left\|\operatorname{div} \mathbf{f}\right\|_p \leq 2C_{\mathbf{g}}^2 \Big\}, \end{aligned}$$

with $C_{\mathbf{g}}$ from (3.82), and E representing the upper bound for the kinetic energy, see (3.69). However, $M_{\mathbf{v}}(m)$ is not a compact subset of $W^{2,p}(\Omega)$ neither a closed subset of $W^{1,\infty}(\Omega)$. Therefore, in order to perform in our last step a simple fixed point argument, we need to introduce additionally another set, which is a closed subset of $W^{1,\infty}(\Omega)$, $M_{\mathbf{v}}(m) \subset M_{\operatorname{div} \mathbf{U}}(m)$, namely

$$\begin{aligned} M_{\operatorname{div} \mathbf{U}}(m) =& \Big\{ \mathbf{f} \in W^{1,\infty}(\Omega, \mathbb{R}^3),\ \mathbf{f} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega, \\ & \left\|\nabla \mathbf{f}\right\|_2 \leq E,\ \left\|\nabla \mathbf{f}\right\|_\infty + \left\|\mathbf{f}\right\|_\infty \leq C_{\mathbf{g}},\ m^{\gamma-1} \left\|\operatorname{div} \mathbf{f}\right\|_p \leq 2C_{\mathbf{g}}^2 \Big\}, \end{aligned}$$

Our general strategy is as follows. First, we fix $\mathbf{U} \in M_{\operatorname{div}\mathbf{U}}(m)$, and $\widetilde{r} \in M_r(m)$ and use the Leray–Schauder, as well as the Banach fixed point theorem to show the existence of a solution $(r, \mathbf{v}) \in M_r(m) \times M_{\mathbf{v}}(m)$ to the following system. We denote $\widetilde{\varrho} = m + \widetilde{r}$.

$$
\begin{aligned}
m \operatorname{div} \mathbf{v} + \operatorname{div}(r\mathbf{v}) &= 0, && (3.27)\\
\widetilde{\varrho}\mathbf{U} \cdot \nabla \mathbf{v} - \operatorname{div}\big(2\widetilde{\varrho}\mathbb{D}(\mathbf{v})\big) + \gamma m^{\gamma-1} \nabla r + \nabla R_m(\widetilde{r}) &= \widetilde{\varrho}\mathbf{g} \text{ in } \Omega, && (3.28)\\
\mathbf{v} \cdot \mathbf{n} = 0, \qquad \mathbf{n} \cdot 2\widetilde{\varrho}\mathbb{D}(\mathbf{v}) \cdot \boldsymbol{\tau}^k + \alpha \mathbf{v} \cdot \boldsymbol{\tau}^k &= 0 \text{ on } \partial\Omega, && (3.29)
\end{aligned}
$$

where, see (3.19),

$$
R_m(\widetilde{r}) = p(m + \widetilde{r}) - m^\gamma - \gamma m^{\gamma-1}\widetilde{r}, \qquad |R_m(\widetilde{r})| \le C m^{\gamma-2}\widetilde{r}^2.
$$

The uniqueness of the solution follows from the construction. Then, still fixing $\mathbf{U} \in M_{\operatorname{div}\mathbf{U}}(m)$, we show via the Banach contraction principle that there exists a solution $(r, \mathbf{v}) \in M_r(m) \times M_{\mathbf{v}}(m)$ to the system

$$
\begin{aligned}
m \operatorname{div} \mathbf{v} + \operatorname{div}(r\mathbf{v}) &= 0, && (3.30)\\
\varrho\mathbf{U} \cdot \nabla \mathbf{v} - \operatorname{div}\big(2\varrho\mathbb{D}(\mathbf{v})\big) + \gamma m^{\gamma-1} \nabla r + \nabla R_m(r) &= \varrho\mathbf{g} \text{ in } \Omega, && (3.31)\\
\mathbf{v} \cdot \mathbf{n} = 0, \qquad \mathbf{n} \cdot 2\varrho\mathbb{D}(\mathbf{v}) \cdot \boldsymbol{\tau}^k + \alpha \mathbf{v} \cdot \boldsymbol{\tau}^k &= 0 \text{ on } \partial\Omega. && (3.32)
\end{aligned}
$$

Finally, we will show the existence of a fixed point of the mapping $\mathcal{T}(\mathbf{U}) = \mathbf{v}$ in $M_{\mathbf{v}}(m)$ by means of the Schauder fixed point theorem.

Existence of solution for fully linearized system

To start, let us show the existence of a unique solution to system (3.27)–(3.29).

Proposition 3.3. *Suppose $\mathbf{U} \in M_{\operatorname{div}\mathbf{U}}(m)$, $\widetilde{\mathbf{v}} \in M_{\mathbf{v}}(m)$, $\widetilde{r} \in M_r(m)$ for m sufficiently large ($C_{\mathbf{g}} \ll m$), then there exists a unique solution $(r, \mathbf{v})$ to problem (3.27)–(3.29) in the class $M_r(m) \times M_{\mathbf{v}}(m)$.*

Proof. First, we study for given $\mathbf{F} \in L^p(\Omega)$ and $h \in W^{1-\frac{1}{p},p}(\partial\Omega)$ the problem

$$
\begin{aligned}
m \operatorname{div} \mathbf{v} + \operatorname{div}(r\widetilde{\mathbf{v}}) &= 0, && (3.33)\\
-m\Delta \mathbf{v} - m\nabla \operatorname{div} \mathbf{v} + \gamma m^{\gamma-1}\nabla r &= \mathbf{F} - \widetilde{\varrho}\mathbf{U} \cdot \nabla \mathbf{v} \text{ in } \Omega, && (3.34)\\
\mathbf{v} \cdot \mathbf{n} &= 0, && (3.35)\\
\mathbf{n} \cdot 2m\mathbb{D}(\mathbf{v}) \cdot \boldsymbol{\tau}^k + \alpha \mathbf{v} \cdot \boldsymbol{\tau}^k &= h \text{ on } \partial\Omega, \quad \int_\Omega r \, \mathrm{d}x = 0. && (3.36)
\end{aligned}
$$

Lemma 3.4. *For given $\mathbf{F} \in L^p(\Omega)$ and $h \in W^{1-\frac{1}{p},p}(\partial\Omega)$, there exists a unique solution to system (3.33)–(3.36) with $r \in W^{1,p}(\Omega)$, $\mathbf{v} \in W^{2,p}(\Omega)$.*

Proof of Lemma. First note that the system is linear. We will proceed in the following way, we fix $\overline{r} \in W^{1,2}(\Omega)$ and use elliptic regularization of the continuity equation in order to get merely weak solution to system with fully linearized continuity equation, then we use the Leray–Schauder argument to obtain solution to (3.33)–(3.36), and finally improve the regularity using the method of decomposition.

For $\varepsilon > 0$ and $\overline{r} \in W^{1,2}(\Omega)$ we consider

$$-\varepsilon\Delta r + \varepsilon r + m \operatorname{div} \mathbf{v} + \operatorname{div}(\overline{r}\widetilde{\mathbf{v}}) = 0, \tag{3.37}$$
$$-m\Delta\mathbf{v} - m\nabla \operatorname{div} \mathbf{v} + \gamma m^{\gamma-1}\nabla r + \widetilde{\varrho}\mathbf{U}\cdot\nabla\mathbf{v} = \mathbf{F} \text{ in } \Omega, \tag{3.38}$$
$$\mathbf{v}\cdot\mathbf{n} = 0, \qquad \mathbf{n}\cdot\nabla r = 0, \tag{3.39}$$
$$\mathbf{n}\cdot 2m\mathbb{D}(\mathbf{v})\cdot\boldsymbol{\tau}^k + \alpha\mathbf{v}\cdot\boldsymbol{\tau}^k = h \text{ on } \partial\Omega. \tag{3.40}$$

It is a strictly elliptic problem, hence the existence of a unique solution follows from the Lax–Milgram theorem; note that $\|\operatorname{div}(\widetilde{\varrho}\mathbf{U})\|_{L^2(\Omega)} \ll m$, so there is no problem with the convective term. Further, we have estimates

$$\varepsilon m^{\gamma-2}\|r\|^2_{W^{1,2}(\Omega)} + m\|\mathbf{v}\|^2_{W^{1,2}(\Omega)} \le C(\mathbf{F}, h, \mathbf{U}, \widetilde{\mathbf{v}}, \widetilde{r}, \|\overline{r}\|_{W^{1,2}(\Omega)}) \tag{3.41}$$

with C independent of ε, and from (3.37) we conclude that actually $r \in W^{2,2}(\Omega)$, see Theorem 4.11. Therefore, we see that the mapping $\mathcal{T} : \overline{r} \mapsto r$ defined through (3.37)–(3.40) is continuous and compact mapping on $W^{1,2}(\Omega)$ for any $\varepsilon > 0$. To apply the Leray–Schauder fixed point theorem 4.9, it remains to show that the possible fixed points

$$\ell\mathcal{T}(r) = r \tag{3.42}$$

are bounded in $W^{1,2}(\Omega)$ independently of $\ell \in [0,1]$. Relation (3.42) is in fact nothing but

$$-\varepsilon\Delta r + \varepsilon r + \ell m \operatorname{div} \mathbf{v} + \ell \operatorname{div}(r\widetilde{\mathbf{v}}) = 0, \tag{3.43}$$
$$-\ell m\Delta\mathbf{v} - \ell m\nabla \operatorname{div} \mathbf{v} + \gamma m^{\gamma-1}\nabla r + \ell\widetilde{\varrho}\mathbf{U}\cdot\nabla\mathbf{v} = \ell\mathbf{F} \text{ in } \Omega, \tag{3.44}$$
$$\mathbf{v}\cdot\mathbf{n} = 0, \qquad \mathbf{n}\cdot\nabla r = 0, \tag{3.45}$$
$$\mathbf{n}\cdot 2m\mathbb{D}(\mathbf{v})\cdot\boldsymbol{\tau}^k + \alpha\mathbf{v}\cdot\boldsymbol{\tau}^k = h \text{ on } \partial\Omega. \tag{3.46}$$

As one could expect, we test the second equation with $\ell\mathbf{v}$ and the first one with $\gamma m^{\gamma-2} r$ yielding

$$\begin{aligned}\ell^2 m\|\mathbf{v}\|^2_{W^{1,2}(\Omega)} &+ \varepsilon m^{\gamma-2}\gamma\|\nabla r\|^2_{L^2(\Omega)} + \varepsilon m^{\gamma-2}\gamma\|r\|^2_{L^2(\Omega)}\\ &\le \ell^2\|\mathbf{F}\|_{L^{6/5}(\Omega)}\|\mathbf{v}\|_{L^6(\Omega)} + \ell m^{\gamma-2}\gamma\|\operatorname{div}\widetilde{\mathbf{v}}\|_{L^\infty(\Omega)}\|r\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.47}$$

In order to close the estimates we need to bound the last term by means of the Bogovskii estimate. This reads,

$$\gamma m^{\gamma-1}\|r\|^2_{L^2(\Omega)} \le C\ell\|r\|_{L^2(\Omega)}\big(m\|\nabla\mathbf{v}\|_{L^2(\Omega)} + m\|\mathbf{U}\|_{L^3(\Omega)}\|\nabla\mathbf{v}\|_{L^2(\Omega)} + \|\mathbf{F}\|_{L^{6/5}(\Omega)}\big),$$

and after using Young's inequality

$$\begin{aligned}&\gamma m^{\gamma-1}\|r\|^2_{L^2(\Omega)}\\ &\quad\le C\ell^2\big(m^{3-\gamma}\|\nabla\mathbf{v}\|^2_{L^2(\Omega)} + m^{3-\gamma}\|\mathbf{U}\|^2_{L^3(\Omega)}\|\nabla\mathbf{v}\|^2_{L^2(\Omega)} + m^{1-\gamma}\|\mathbf{F}\|^2_{L^{6/5}(\Omega)}\big).\end{aligned}$$

Incorporating this into (3.47), we obtain

$$\begin{aligned}&\ell^2 m\|\mathbf{v}\|^2_{W^{1,2}(\Omega)} + \varepsilon m^{\gamma-2}\|\nabla r\|^2_{L^2(\Omega)} + \varepsilon m^{\gamma-2}\|r\|^2_{L^2(\Omega)}\\ &\quad\le C\left(\frac{\ell^2}{m}\|\mathbf{F}\|^2_{L^{6/5}(\Omega)} + \frac{\ell^3}{m}\|\operatorname{div}\widetilde{\mathbf{v}}\|_{L^\infty(\Omega)}\Big(\frac{\|\nabla\mathbf{v}\|^2_{L^2(\Omega)}}{m^{\gamma-3}}(1+E^2) + \frac{\|\mathbf{F}\|^2_{L^{6/5}(\Omega)}}{m^{\gamma-1}}\Big)\right)\end{aligned}$$

and consequently, since $\|\nabla^2\widetilde{\mathbf{v}}\|_{L^p(\Omega)} \ll m^{\gamma-1}$,

$$m^{\gamma-1}\|r\|_{L^2(\Omega)} + \ell m\|\mathbf{v}\|_{W^{1,2}(\Omega)} \leq C\big(\|\mathbf{F}\|_{L^{6/5}(\Omega)}\big), \tag{3.48}$$

where C is independent of ε and ℓ. Thus, we get for any $\varepsilon > 0$ a fixed point of $\mathcal{T}$, which satisfies (3.48) with $\ell = 1$, hence we can pass to the limit with $\varepsilon \to 0^+$ to get a weak solution to (3.33)–(3.36).

To improve the regularity of the solution we use the method of decomposition of Novotný and Padula [113]. First, we deduce by applying curl on (3.34) that

$$\begin{aligned}
-m\Delta\boldsymbol{\omega} &= \operatorname{curl}\big(-\widetilde{\varrho}\mathbf{U}\cdot\nabla\mathbf{v} + \mathbf{F}\big) \text{ in } \Omega,\\
\boldsymbol{\omega}\cdot\boldsymbol{\tau}^1 &= -\Big(2\chi_2 - \frac{\alpha}{m}\Big)\mathbf{v}\cdot\boldsymbol{\tau}^2 - \frac{h}{m},\\
\boldsymbol{\omega}\cdot\boldsymbol{\tau}^2 &= \Big(2\chi_1 - \frac{\alpha}{m}\Big)\mathbf{v}\cdot\boldsymbol{\tau}^1 + \frac{h}{m},\\
\operatorname{div}\boldsymbol{\omega} &= 0 \text{ on } \partial\Omega,
\end{aligned}$$

so

$$\begin{aligned}
m\|\boldsymbol{\omega}\|_{W^{1,p}(\Omega)} \leq C\Big(&\|\operatorname{curl}(-\widetilde{\varrho}\mathbf{U}\cdot\nabla\mathbf{v} + \mathbf{F})\|_{(W^{1,p'}(\Omega))^*}\\
&+ m\|\mathbf{v}\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} + \|h\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}\Big),
\end{aligned} \tag{3.49}$$

and as $P_H\mathbf{v}$ satisfies (3.16), we get by Theorem 4.14 that

$$\begin{aligned}
m\left\|\nabla^2 P_H\mathbf{v}\right\|_{L^p(\Omega)} \leq C\Big(&m\|\nabla\mathbf{v}\|_{L^p(\Omega)} + \|\mathbf{F}\|_{L^p(\Omega)}\\
&+ m\|\mathbf{v}\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} + \|h\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}\Big).
\end{aligned} \tag{3.50}$$

Further, using the well-known vector identity

$$\Delta\mathbf{v} = \nabla\operatorname{div}\mathbf{v} - \operatorname{curl}(\operatorname{curl}\mathbf{v}), \tag{3.51}$$

we observe that the linearized effective viscous flux

$$G = \gamma m^{\gamma-2} r - 2\operatorname{div}\mathbf{v} \tag{3.52}$$

solves

$$m\nabla G = \mathbf{F} - \widetilde{\varrho}\mathbf{U}\cdot\nabla\mathbf{v} - m\operatorname{curl}\boldsymbol{\omega}, \qquad \int_\Omega G\,\mathrm{d}x = 0, \tag{3.53}$$

with the estimate

$$m\|G\|_{W^{1,p}(\Omega)} \leq C\big(\|\mathbf{F}\|_{L^p(\Omega)} + m\|\nabla\mathbf{v}\|_{L^p(\Omega)} + m\|\operatorname{curl}\boldsymbol{\omega}\|_{L^p(\Omega)}\big). \tag{3.54}$$

Next, combining the continuity equation (3.33) together with relation (3.52), we observe that the variation of the density r actually satisfies the stationary transport equation

$$r + \operatorname{div}\Big(\frac{r\widetilde{\mathbf{v}}}{\gamma m^{\gamma-1}}\Big) = \frac{2G}{\gamma m^{\gamma-2}} \text{ in } \Omega, \qquad \int_\Omega r\,\mathrm{d}x = 0. \tag{3.55}$$

Noting that

$$\frac{\|\widetilde{\mathbf{v}}\|_{W^{2,p}(\Omega)}}{m^{\gamma-1}} \le \alpha \tag{3.56}$$

for some α sufficiently small and $\widetilde{\mathbf{v}} \cdot \mathbf{n} = 0$ on $\partial\Omega$, we can deduce that the unique solution r of problem (3.55) satisfies

$$m^{\gamma-2} \|r\|_{W^{1,p}(\Omega)} \le C \|G\|_{W^{1,p}(\Omega)},$$

see [118, Lemma 5.11].

Finally, the definition of Helmholtz decomposition yields that actually we have $\operatorname{div} \mathbf{v} = \Delta P_\triangledown \mathbf{v}$, hence according to (3.52) the potential part of the velocity field $P_\triangledown \mathbf{v}$ satisfies the Neumann problem

$$-2\Delta P_\triangledown \mathbf{v} = G - \gamma m^{\gamma-2} r \text{ in } \Omega, \tag{3.57}$$

$$\nabla P_\triangledown \mathbf{v} \cdot \mathbf{n} = \ 0 \text{ on } \partial\Omega, \tag{3.58}$$

providing by Theorem 4.11 the estimate

$$m \|\nabla P_\triangledown \mathbf{v}\|_{W^{2,p}(\Omega)} \le Cm \left\|G - \gamma m^{\gamma-2} r\right\|_{W^{1,p}(\Omega)}.$$

Therefore, summing up the estimates above, we get that solution to (3.43)–(3.46) fulfils

$$\begin{aligned} m \|\mathbf{v}\|_{W^{2,p}(\Omega)} + m^{\gamma-1} \|r\|_{W^{1,p}(\Omega)} \le C\Big(& m \|\nabla \mathbf{v}\|_{L^p(\Omega)} + m \|\mathbf{v}\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \\ & + \|\mathbf{F}\|_{L^p(\Omega)} + \|h\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}\Big). \end{aligned}$$

The first two terms can be put to the left-hand side by means of interpolation with the energy norm, while the rest is controlled, so we see that the solution has the proposed regularity. This completes the proof of this lemma. □

In order to finish the proof of Proposition 3.3 we will find a fixed point of the mapping $\widetilde{\mathbf{v}} \mapsto \mathbf{v}$ defined through[2]

$$m \operatorname{div} \mathbf{v} + \operatorname{div}(r\widetilde{\mathbf{v}}) = 0, \tag{3.59}$$

$$-\operatorname{div}\big(2m\mathbb{D}(\mathbf{v})\big) + \gamma m^{\gamma-1}\nabla r = \operatorname{div}\big(2\widetilde{r}\mathbb{D}(\widetilde{\mathbf{v}})\big) + \nabla R_m(\widetilde{r}) + \widetilde{\varrho}\mathbf{g} - \widetilde{\varrho}\mathbf{U} \cdot \nabla \mathbf{v} \text{ in } \Omega, \tag{3.60}$$

$$\mathbf{v} \cdot \mathbf{n} = 0, \tag{3.61}$$

$$\mathbf{n} \cdot 2m\mathbb{D}(\mathbf{v}) \cdot \boldsymbol{\tau}^k + \alpha \mathbf{v} \cdot \boldsymbol{\tau}^k = -\mathbf{n} \cdot 2\widetilde{r}\mathbb{D}(\widetilde{\mathbf{v}}) \cdot \boldsymbol{\tau}^k \text{ on } \partial\Omega, \quad \int_\Omega r \,\mathrm{d}x = 0. \tag{3.62}$$

The mapping is according to the previous lemma well-defined from $W^{2,p}(\Omega)$ to $W^{2,p}(\Omega)$. We want to show that in fact it maps $M_{\mathbf{v}}(m)$ into itself and that it is a contraction. For this purpose, we test the first equation with $\gamma m^{\gamma-2} r$, the second equation with $\mathbf{v}$, and sum up the resulting relations. We end up with

$$\begin{aligned} \int_\Omega 2m \left|\mathbb{D}(\mathbf{v})\right|^2 \mathrm{d}x + \sum_{k=1,2} \int_{\partial\Omega} \alpha \left|\mathbf{v} \cdot \boldsymbol{\tau}^k\right|^2 \mathrm{d}S = \int_\Omega 2\widetilde{r}\mathbb{D}(\widetilde{\mathbf{v}}) : \mathbb{D}(\mathbf{v}) \,\mathrm{d}x \\ + \int_\Omega \Big(-\widetilde{\varrho}\mathbf{U} \cdot \nabla \frac{|\mathbf{v}|^2}{2} + \big(-\frac{\gamma m^{\gamma-2}}{2} r^2 \operatorname{div} \widetilde{\mathbf{v}} + R_m(\widetilde{r}) \operatorname{div} \mathbf{v} + \widetilde{\varrho}\mathbf{g} \cdot \mathbf{v}\big)\Big) \,\mathrm{d}x. \end{aligned} \tag{3.63}$$

[2]Let us recall the notation $\widetilde{\varrho} = m + \widetilde{r}$.

The first term on the right-hand side can be put directly to the left-hand side while the convective term can be estimated

$$\left|\int_\Omega \operatorname{div}\big((m+\widetilde{r})\mathbf{U}\big)\frac{|\mathbf{v}|^2}{2}\,\mathrm{d}x\right| \le m\int_\Omega |\operatorname{div}\mathbf{U}|\,|\mathbf{v}|^2\,\mathrm{d}x + \int_\Omega |\nabla\widetilde{r}|\,|\mathbf{U}|\,|\mathbf{v}|^2\,\mathrm{d}x$$
$$\le C\,\|\mathbf{v}\|^2_{W^{1,2}(\Omega)}\Big(m\,\|\operatorname{div}\mathbf{U}\|_{L^p(\Omega)} + \|\nabla\widetilde{r}\|_{L^p(\Omega)}\,\|\mathbf{U}\|_{L^3(\Omega)}\Big). \quad (3.64)$$

Thus, using the assumptions on $\widetilde{r},\mathbf{U}$, especially $C^2_{\mathbf{g}} \ll \min(m, m^{\gamma-1})$

$$m\big\|\nabla\mathbf{v}\big\|_2^2 \le C\Big(m^{\gamma-2}\,\|r\|_2^2\,\big\|\operatorname{div}\widetilde{\mathbf{v}}\big\|_\infty + m\big\|\mathbf{g}\big\|^2_{6/5} + 1\Big). \quad (3.65)$$

In order to obtain the L^2-estimate of the density, we will now test the momentum equation with $-\mathbf{\Phi}$, $\mathbf{\Phi} = \mathcal{B}[r]$, so $\|\nabla\mathbf{\Phi}\|_2 \le C\,\|r\|_2$. This leads to

$$\gamma m^{\gamma-1}\,\|r\|_2^2 \le m^{\gamma-2}\,\|\widetilde{r}\|_\infty\,\|\widetilde{r}\|_2\,\|r\|_2 + 2m\,\|\nabla\mathbf{v}\|_2\,\|\nabla\mathbf{\Phi}\|_2 + 2\,\|\widetilde{r}\|_\infty\,\|\nabla\widetilde{\mathbf{v}}\|_2\,\|\nabla\mathbf{\Phi}\|_2$$
$$+ \int_\Omega \Big((m+\widetilde{r})\mathbf{U}\cdot\nabla\mathbf{v}\cdot\mathbf{\Phi} - (m+\widetilde{r})\mathbf{g}\cdot\mathbf{\Phi}\Big)\,\mathrm{d}x$$
$$\le C\Big(\big(m^{\gamma-2}\big\|\widetilde{r}\big\|_2 + \|\nabla\widetilde{\mathbf{v}}\|_2\big)\,\|\widetilde{r}\|_\infty + m\,\|\nabla\mathbf{v}\|_2 + m\,\|\mathbf{U}\|_6\,\|\nabla\mathbf{v}\|_2 + m\,\|\mathbf{g}\|_{6/5}\Big)\,\|r\|_2\,,$$

id est

$$m^{\gamma-1}\,\|r\|_2^2 \le C\Big(m^{\gamma-3}\|\widetilde{r}\|^2_\infty\,\|\widetilde{r}\|_2^2 + m^{3-\gamma}\big(\|\nabla\mathbf{v}\|_2^2 + \|\nabla\mathbf{v}\|_2^2\,E^2 + \|\mathbf{g}\|^2_{6/5}\big)\Big), \quad (3.66)$$

$$m^{\gamma-1}\,\|r\|_2^2 \le C\Big(m^{\gamma-3}\|\widetilde{r}\|^2_\infty\,\|\widetilde{r}\|_2^2 + (1+E^2)\big(m^{3-\gamma}\,\|\mathbf{g}\|^2_{6/5} + 1 + \|r\|_2^2\,\|\operatorname{div}\widetilde{\mathbf{v}}\|_\infty\big)\Big). \quad (3.67)$$

Assuming $m^{\gamma-1} \gg E^2 C_{\mathbf{g}}$, the last term can be put to the left-hand side, hence going back to (3.65), we obtain

$$m\big\|\nabla\mathbf{v}\big\|_2^2$$
$$\le C\left(\left(\frac{\big\|\widetilde{r}\big\|^2_\infty\big\|\widetilde{r}\big\|^2_2}{m^{4-\gamma}} + (1+E^2)\Big(\frac{\|\mathbf{g}\|^2_{\frac{6}{5}}}{m^{\gamma-2}} + \frac{1}{m^{\gamma-1}}\Big)\right)\big\|\operatorname{div}\widetilde{\mathbf{v}}\big\|_\infty + m\,\|\mathbf{g}\|^2_{\frac{6}{5}} + 1\right),$$

$$\big\|\nabla\mathbf{v}\big\|_2^2 \le C\left(\left(\frac{m^{7-3\gamma}}{m^{5-\gamma}} + (1+E^2)\Big(\frac{\|\mathbf{g}\|^2_{\frac{6}{5}}}{m^{\gamma-1}} + \frac{1}{m^\gamma}\Big)\right)\big\|\operatorname{div}\widetilde{\mathbf{v}}\big\|_\infty + \|\mathbf{g}\|^2_{\frac{6}{5}} + m^{-1}\right).$$

This yields

$$\|\nabla\mathbf{v}\|_{L^2(\Omega)} \le E, \quad (3.68)$$

provided E, m are such that

$$C_1\left[\left(m^{2-2\gamma} + \frac{1}{m} + (1+E^2)\Big(\frac{\|\mathbf{g}\|^2_{\frac{6}{5}}}{m^{\gamma-1}} + \frac{1}{m^\gamma}\Big)\right)m^{\frac{(\gamma-1)}{2}} + \|\mathbf{g}\|^2_{6/5}\right] \le E^2. \quad (3.69)$$

Moreover,

$$m^{\gamma-2}\,\|r\|_2 \le C_2\big(\big\|\mathbf{g}\big\|_{6/5} + \big\|\mathbf{g}\big\|^2_{6/5}\big). \quad (3.70)$$

Next we will show that for $(r, \mathbf{v})$ we have $\Xi \leq C_{\mathbf{g}}$ as well. This will closely follow the a priori estimates. Let us denote

$$\widetilde{\mathcal{G}} = -\widetilde{\varrho}\mathbf{U} \cdot \nabla \mathbf{v} + 2\mathbb{D}(\widetilde{\mathbf{v}})\nabla \widetilde{r} + \widetilde{r}\Delta\widetilde{\mathbf{v}} + \widetilde{r}\nabla \operatorname{div} \widetilde{\mathbf{v}} + \widetilde{\varrho}\mathbf{g},$$

where $\widetilde{\varrho} = \widetilde{r} + m$. First, applying curl on (3.60) yields

$$\begin{aligned} -m\Delta\boldsymbol{\omega} &= \operatorname{curl}\widetilde{\mathcal{G}} \text{ in } \Omega, && (3.71)\\ m\boldsymbol{\omega}\cdot\boldsymbol{\tau}^1 &= -\widetilde{r}\widetilde{\boldsymbol{\omega}}\cdot\boldsymbol{\tau}^1 - \Big(2m\chi_2 - \alpha\Big)\mathbf{v}\cdot\boldsymbol{\tau}^2 - 2\widetilde{r}\chi_2\widetilde{\mathbf{v}}\cdot\boldsymbol{\tau}^2, && (3.72)\\ m\boldsymbol{\omega}\cdot\boldsymbol{\tau}^2 &= -\widetilde{r}\widetilde{\boldsymbol{\omega}}\cdot\boldsymbol{\tau}^2 + \Big(2m\chi_1 - \alpha\Big)\mathbf{v}\cdot\boldsymbol{\tau}^1 + 2\widetilde{r}\chi_2\widetilde{\mathbf{v}}\cdot\boldsymbol{\tau}^1, && (3.73)\\ \operatorname{div}\boldsymbol{\omega} &= 0 \text{ on } \partial\Omega, && (3.74) \end{aligned}$$

and since $P_H\mathbf{v}$ satisfies (3.16), we conclude

$$m\left\|\nabla^2 P_H\mathbf{v}\right\|_p \leq C\Big(\|\nabla(\widetilde{r}\widetilde{\boldsymbol{\omega}})\|_p + \big\|\widetilde{\mathcal{G}}\big\|_{L^p(\Omega)} + m\|\nabla\mathbf{v}\|_p + \|\nabla(\widetilde{r}\widetilde{\mathbf{v}})\|_p\Big). \quad (3.75)$$

Similarly, the potential part of the momentum equation (3.60) reads

$$\gamma m^{\gamma-1} r + R_m(\widetilde{r}) - \{R_m(\widetilde{r})\}_\Omega - 2m\operatorname{div}\mathbf{v} = P_\nabla\big(m\Delta P_H\mathbf{v} + \widetilde{\mathcal{G}}\big) \quad (3.76)$$

which combined with the continuity equation

$$m\operatorname{div}\mathbf{v} + \operatorname{div}(r\widetilde{\mathbf{v}}) = 0$$

yields

$$\gamma m^{\gamma-1} r + R_m(\widetilde{r}) - \{R_m(\widetilde{r})\}_\Omega + 2\nabla r\cdot\widetilde{\mathbf{v}} = -2r\operatorname{div}\widetilde{\mathbf{v}} + P_\nabla\big(m\Delta P_H\mathbf{v} + \widetilde{\mathcal{G}}\big).$$

After differentiating,

$$\begin{aligned} &\gamma m^{\gamma-1}\nabla r + 2\widetilde{\mathbf{v}}\cdot\nabla\nabla r\\ &= -2\nabla r\operatorname{div}\widetilde{\mathbf{v}} - 2r\nabla\operatorname{div}\widetilde{\mathbf{v}} - 2\nabla\widetilde{\mathbf{v}}\nabla r - \nabla R_m(\widetilde{r}) + \nabla P_\nabla\big(m\Delta P_H\mathbf{v} + \widetilde{\mathcal{G}}\big). \quad (3.77) \end{aligned}$$

Using the same trick as in the a priori estimates part,

$$\int_\Omega \widetilde{\mathbf{v}}\cdot\nabla\partial_k r\,|\partial_k r|^{p-2}\,\partial_k r\,\mathrm{d}x = -\frac{1}{p}\int_\Omega \operatorname{div}\widetilde{\mathbf{v}}\,|\partial_k r|^p\,\mathrm{d}x,$$

we obtain

$$\begin{aligned} m^{\gamma-1}\|\nabla r\|_p \leq C\Big(&\big\|\nabla\widetilde{\mathbf{v}}\big\|_\infty\big\|\nabla r\big\|_p + \big\|r\big\|_\infty\big\|\nabla\operatorname{div}\widetilde{\mathbf{v}}\big\|_p + \big\|\nabla r\big\|_p\big\|\operatorname{div}\widetilde{\mathbf{v}}\big\|_\infty\\ &+ \|\nabla R_m(\widetilde{r})\|_p + \big\|\widetilde{\mathcal{G}}\big\|_p + m\big\|\nabla^2 P_H\mathbf{v}\big\|_p\Big), \end{aligned}$$

hence since $m^{\gamma-1} \gg \Xi$,

$$\|\nabla r\|_p \leq \frac{C}{m^{\gamma-1}}\Big(\|\nabla R_m(\widetilde{r})\|_p + \big\|\widetilde{\mathcal{G}}\big\|_p + m\big\|\nabla^2 P_H\mathbf{v}\big\|_p\Big). \quad (3.78)$$

Moreover, using (3.76), we can bound the potential part of the velocity. As

$$2m\nabla\operatorname{div}\mathbf{v} = \gamma m^{\gamma-1}\nabla r + \nabla R_m(\widetilde{r}) - \nabla P_\nabla\big(m\Delta P_H\mathbf{v} + \widetilde{\mathcal{G}}\big), \quad (3.79)$$

we obtain from the estimates above

$$\Xi \leq \frac{C}{m}\Big(\|\nabla R_m(\widetilde{r})\|_p + \|\nabla(\widetilde{r}\widetilde{\boldsymbol{\omega}})\|_p + \big\|\widetilde{\mathcal{G}}\big\|_{L^p(\Omega)} + m\,\|\nabla \mathbf{v}\|_p + \|\nabla(\widetilde{r}\widetilde{\mathbf{v}})\|_p\Big). \quad (3.80)$$

According to the fact that $C_{\mathbf{g}}^2 \ll m$, the only problematic term in $\widetilde{\mathcal{G}}$ is again the convective term. At this point we use the fact that $\mathbf{U}$ satisfies the energy inequality; hence

$$\begin{aligned}\big\|\widetilde{\varrho}\mathbf{U}\cdot\nabla\mathbf{v}\big\|_p &\leq \big\|\widetilde{\varrho}\big\|_\infty \big\|\mathbf{U}\big\|_6 \big\|\nabla\mathbf{v}\big\|_{\frac{6p}{6-p}} \\ &\leq \big(m + \big\|\widetilde{r}\big\|_\infty\big)\big\|\mathbf{U}\big\|_6 \big\|\nabla\mathbf{v}\big\|_2^{\frac{6-p}{3p}} \big\|\nabla\mathbf{v}\big\|_\infty^{\frac{4p-6}{3p}} \leq CmE^{\frac{2p+6}{3p}}\Xi^{\frac{4p-6}{3p}}.\end{aligned}$$

Thus,

$$\Xi \leq C\Big(1 + \big\|\mathbf{g}\big\|_p + E\Xi^{\frac{4p-6}{3p}}\big\|\mathbf{g}\big\|_{6/5}^{\frac{2p+6}{3p}}\Big). \quad (3.81)$$

As $\frac{4p-6}{3p} < 1$ for $p < 6$, we conclude finally

$$\begin{aligned} m^{\gamma-2}\big(\|r\|_{W^{1,p}(\Omega)} + \|r\|_{L^\infty(\Omega)}\big) + \|\mathbf{v}\|_{W^{2,p}(\Omega)} + \|\nabla\mathbf{v}\|_{L^\infty(\Omega)} + \|\mathbf{v}\|_{L^\infty(\Omega)} \\ \leq C\Big(1 + \|\mathbf{g}\|_{L^p(\Omega)} + \|\mathbf{g}\|_{L^{6/5}(\Omega)}^{\frac{\frac{2p+6}{3p}}{\frac{6-p}{3p}}} E^{\frac{3p}{6-p}}\Big), \quad (3.82)\end{aligned}$$

where C is an absolute constant independent of the solution, provided $\Xi \ll m$. It is sufficient to choose m to be appropriately greater than the right-hand side of (3.82) — let us denote it by $C_{\mathbf{g}}$. Having in mind that we need to satisfy the restriction (3.69) as well, we take

$$\frac{\min(m, m^{\frac{\gamma-1}{4}})}{\alpha^{-1} + 15} > \max\big(C_{\mathbf{g}}, C_{\mathbf{g}}^2, C_{\mathbf{g}}E^2, C_{\mathbf{g}}^2E^2, C_1, C_2\big)\cdot\max(C_P, C_K, C_E, C_B), \quad (3.83)$$

where C_1 is from (3.90), C_2 from (3.97), α represents the smallness constant in (3.56), and C_P, C_K and C_E denotes the constant from the Poincaré, Korn, and the embedding ($W^{1,p} \hookrightarrow L^\infty$) inequality, respectively. The symbol C_B stands for the constant induced by the Bogovskii operator from Theorem 4.13. Finally, looking back to the continuity equation, we can conclude from estimate (3.82) that actually $\|\operatorname{div}\mathbf{v}\|_{L^p(\Omega)} \leq 2C_{\mathbf{g}}^2/m^{\gamma-1}$.

Now let us prove that mapping $\widetilde{\mathbf{v}} \mapsto \mathbf{v}$ is in fact a contraction. Indeed, we have for difference of two solutions $\mathbf{V} = \mathbf{v}_1 - \mathbf{v}_2$, $R = r_1 - r_2$ corresponding to $\widetilde{\mathbf{V}} = \widetilde{\mathbf{v}}_1 - \widetilde{\mathbf{v}}_2$

$$m\operatorname{div}\mathbf{V} + \operatorname{div}(R\widetilde{\mathbf{v}}_1) + \operatorname{div}(r_2\widetilde{\mathbf{V}}) = 0, \quad (3.84)$$

$$\widetilde{\varrho}\mathbf{U}\cdot\nabla\mathbf{V} - 2m\operatorname{div}\big(\mathbb{D}(\mathbf{V})\big) - \operatorname{div}\big(2\widetilde{r}\mathbb{D}(\widetilde{\mathbf{V}})\big) + \gamma m^{\gamma-1}\nabla R = \mathbf{0} \text{ in } \Omega, \quad (3.85)$$

$$\mathbf{V}\cdot\mathbf{n} = 0, \quad \mathbf{n}\cdot 2m\mathbb{D}(\mathbf{V})\cdot\boldsymbol{\tau}^k + \alpha\mathbf{V}\cdot\boldsymbol{\tau}^k = -\mathbf{n}\cdot 2\widetilde{r}\mathbb{D}(\widetilde{\mathbf{V}})\cdot\boldsymbol{\tau}^k \text{ on } \partial\Omega. \quad (3.86)$$

Basic energy estimate reads

$$\begin{aligned}\int_\Omega 2m\,|\mathbb{D}(\mathbf{V})|^2\,\mathrm{d}x + \sum_{k=1}^2\int_{\partial\Omega}\alpha\,\big|\mathbf{V}\cdot\boldsymbol{\tau}^k\big|^2\,\mathrm{d}S = \int_\Omega 2\widetilde{r}\mathbb{D}(\widetilde{\mathbf{V}}) : \mathbb{D}(\mathbf{V})\,\mathrm{d}x \\ = \int_\Omega\Big(-\widetilde{\varrho}\mathbf{U}\cdot\nabla\frac{|\mathbf{V}|^2}{2} - \frac{\gamma m^{\gamma-2}}{2}R^2\operatorname{div}\widetilde{\mathbf{v}}_1 + \operatorname{div}(r_2\widetilde{\mathbf{V}})\gamma m^{\gamma-2}R\Big)\,\mathrm{d}x. \quad (3.87)\end{aligned}$$

Further, using again (3.64), we obtain

$$m\|\nabla\mathbf{V}\|_2^2 \le C\Big(m^{\gamma-2}\|R\|_2^2\|\operatorname{div}\widetilde{\mathbf{v}}_1\|_\infty + \frac{\|\widetilde{r}\|_\infty^2}{m}\big\|\nabla\widetilde{\mathbf{V}}\big\|_2^2 + m^{\gamma-2}\big\|\operatorname{div}(r_2\widetilde{\mathbf{V}})\big\|_2\|R\|_2\Big). \tag{3.88}$$

Estimating the density by means of the Bogovskii operator leads to

$$\begin{aligned}\gamma m^{\gamma-1}\|R\|_2^2 &\le 2m\|\nabla\mathbf{V}\|_2\|\nabla\mathbf{\Phi}\|_2 + \|\widetilde{r}\|_\infty\left\|\nabla\widetilde{\mathbf{V}}\right\|_2\|\nabla\mathbf{\Phi}\|_2\\ &\qquad + \int_\Omega\Big((m+\widetilde{r})\mathbf{U}\cdot\nabla\mathbf{V}\cdot\mathbf{\Phi}\Big)\,\mathrm{d}x\\ &\le C\Big(m\|\nabla\mathbf{V}\|_2 + \|\widetilde{r}\|_\infty\left\|\nabla\widetilde{\mathbf{V}}\right\|_2 + m\|\mathbf{U}\|_6\|\nabla\mathbf{V}\|_2\Big)\|R\|_2,\end{aligned}$$

hence by Young's inequality and (3.88)

$$\begin{aligned}m^{\gamma-1}\|R\|_2^2 &\le C\Big(m^{3-\gamma}\|\nabla\mathbf{V}\|_2^2 + \frac{\|\widetilde{r}\|_\infty^2\big\|\nabla\widetilde{\mathbf{V}}\big\|_2^2}{m^{\gamma-1}} + m^{3-\gamma}E^2\|\nabla\mathbf{V}\|_2^2\Big)\\ &\le C(1+E^2)\Big(\|R\|_2^2\|\operatorname{div}\widetilde{\mathbf{v}}_1\|_\infty + \big\|\operatorname{div}(r_2\widetilde{\mathbf{V}})\big\|_2\|R\|_2\Big) + C\frac{\|\widetilde{r}\|_\infty^2}{m^{\gamma-1}}\big\|\nabla\widetilde{\mathbf{V}}\big\|_2^2.\end{aligned}$$

As $\|\operatorname{div}\widetilde{\mathbf{v}}_1\|_\infty \ll m^{\gamma-1}$ the first term can be put to the left-hand side, so we get again by Young's inequality

$$m^{\gamma-1}\|R\|_2^2 \le C\left((1+E^4)\frac{\big\|\operatorname{div}(r_2\widetilde{\mathbf{V}})\big\|_2^2}{m^{\gamma-1}} + \frac{\|\widetilde{r}\|_\infty^2}{m^{\gamma-1}}\big\|\nabla\widetilde{\mathbf{V}}\big\|_2^2\right).$$

Alternatively, since $\big\|\operatorname{div}(r_2\widetilde{\mathbf{V}})\big\|_2 \le \|r_2\|_\infty\big\|\nabla\widetilde{\mathbf{V}}\big\|_2 + \|\nabla r_2\|_3\big\|\widetilde{\mathbf{V}}\big\|_6$, we can write

$$m^{\gamma-1}\|R\|_2 \le C(1+E^2)C_{\mathbf{g}}\big\|\nabla\widetilde{\mathbf{V}}\big\|_2.$$

so going back to (3.88)

$$m\|\nabla\mathbf{V}\|_2^2 \le C\big(1+E^2\big)\Big(\frac{C_{\mathbf{g}}^2}{m^\gamma}\big\|\nabla\widetilde{\mathbf{V}}\big\|_2^2\|\operatorname{div}\widetilde{\mathbf{v}}_1\|_\infty + \frac{C_{\mathbf{g}}^2}{m}\big\|\nabla\widetilde{\mathbf{V}}\big\|_2^2\Big). \tag{3.89}$$

Therefore, for m sufficiently large, we can write for some C_1, which is independent of m,

$$\|\nabla\mathbf{V}\|_2 \le \frac{C_1}{m}\big\|\nabla\widetilde{\mathbf{V}}\big\|_2. \tag{3.90}$$

Taking $m > C_1$ we obtain that the mapping is contraction in the $W^{1,2}$-metric. Thus, applying Theorem 4.10 on set $M_{\mathbf{v}}(m) \subset W^{2,p}(\Omega)$ yields the result. □

Elimination of the density linearization

Proposition 3.5. *Suppose $\mathbf{U} \in M_{\mathbf{v}}(m)$ for m sufficiently large, then there exists a unique solution $(r, \mathbf{v})$ to problem (3.30)–(3.31) in the class $M_r(m) \times M_{\mathbf{v}}(m)$.*

Proof. We will apply Theorem 4.10 on the mapping

$$\mathcal{S}_{\mathbf{U}} : M_r(m) \to M_r(m),$$

defined as a solution operator to the following problem $\mathcal{S}(r_n) = r_{n+1}$

$$m \operatorname{div} \mathbf{v}_{n+1} + \operatorname{div}(r_{n+1}\mathbf{v}_{n+1}) = 0, \tag{3.91}$$

$$(m + r_n)\, \mathbf{U} \cdot \nabla \mathbf{v}_{n+1} - \operatorname{div}\big(2(m + r_n)\mathbb{D}(\mathbf{v}_{n+1})\big) + \gamma m^{\gamma-1} \nabla r_{n+1} + \nabla R_m(r_n) = (m + r_n)\mathbf{g} \text{ in } \Omega, \tag{3.92}$$

$$\mathbf{v}_{n+1} \cdot \mathbf{n} = 0,$$
$$\mathbf{n} \cdot 2(m + r_n)\mathbb{D}(\mathbf{v}_{n+1}) \cdot \boldsymbol{\tau}^k + \alpha \mathbf{v}_{n+1} \cdot \boldsymbol{\tau}^k = 0 \text{ on } \partial\Omega, \quad \int_\Omega r_{n+1} \, \mathrm{d}x = 0. \tag{3.93}$$

The solvability of system (3.92) in $M_r(m) \times M_{\mathbf{v}}(m)$ was proven in Proposition 3.3. Thus, $\mathcal{S}$ indeed maps $M_r(m)$ into itself.

We will show that $\mathcal{S}$ is contraction. Let us denote

$$\mathbf{v} = \mathbf{v}_{n+1} - \mathbf{v}_n, \; r = r_{n+1} - r_n, \; r_- = r_n - r_{n-1},$$

then the difference $(\mathbf{v}, r)$ satisfies

$$\begin{aligned} m \operatorname{div} \mathbf{v} + \operatorname{div}(r\mathbf{v}_{n+1}) + \operatorname{div}(r_n \mathbf{v}) = 0 \\ (m + r_n)\mathbf{U} \cdot \nabla \mathbf{v} + r_- \mathbf{U} \cdot \nabla \mathbf{v}_n - \operatorname{div}(2(m + r_n)\mathbb{D}(\mathbf{v})) \\ - \operatorname{div}(2r_- \mathbb{D}(\mathbf{v}_n)) + \gamma m^{\gamma-1}\nabla r + m^{\gamma-2}\nabla\big((r_n + r_{n-1})(r_-)\big) = r_- \mathbf{g} \text{ in } \Omega, \\ \mathbf{v} \cdot \mathbf{n} = 0, \\ \mathbf{n} \cdot 2(m + r_n)\mathbb{D}(\mathbf{v}) \cdot \boldsymbol{\tau}^k + \mathbf{n} \cdot 2r_- \mathbb{D}(\mathbf{v}_n) \cdot \boldsymbol{\tau}^k + \alpha \mathbf{v} \cdot \boldsymbol{\tau}^k = 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.94}$$

First, let us test the momentum equation of (3.94) by the difference $\mathbf{v}$ and the continuity equation by $\gamma m^{\gamma-2} r$, this turns after usage of Hölder's and Young's inequalities into

$$\begin{aligned} m\big\|\nabla \mathbf{v}\big\|_2^2 \le C\bigg(\frac{\|r_-\|_2^2}{m} \|\nabla \mathbf{v}_n\|_\infty^2 (1 + E^2) + m^{\gamma-2} \|r_n + r_{n+1}\|_\infty \|r_-\|_2 \|\operatorname{div} \mathbf{v}\|_2 \\ + m^{\gamma-2}\Big(\|\operatorname{div} \mathbf{v}_{n+1}\|_\infty \|r\|_2^2 + \|\operatorname{div}(r_n \mathbf{v})\|_2 \|r\|_2 \Big) + \frac{\|r_-\|_2^2 \|\mathbf{g}\|_3^2}{m} \bigg). \end{aligned}$$

The second term on the right-hand side can be put directly to the left-hand side, and similarly we proceed with the other term containing $\mathbf{v}$, this leads to

$$\begin{aligned} m\big\|\nabla \mathbf{v}\big\|_2^2 \le C\bigg(\frac{\|r_-\|_2^2}{m} \|\nabla \mathbf{v}_n\|_\infty^2 (1 + E^2) + m^{\gamma-2} \|\operatorname{div} \mathbf{v}_{n+1}\|_\infty \|r\|_2^2 \\ + \frac{\|r_n\|_\infty + \|\nabla r_n\|_3}{m^{5-2\gamma}} \|r\|_2^2 + \frac{\|r_-\|_2^2 \|\mathbf{g}\|_3^2}{m} \bigg). \end{aligned} \tag{3.95}$$

Further, using the Bogovskii type of estimates we obtain

$$\begin{aligned} m^{\gamma-1} \|r\|_2^2 \le C\Big(m \|\mathbf{U}\|_3 \|\nabla \mathbf{v}\|_2 + \|r_-\|_2 \|\mathbf{U}\|_3 \|\nabla \mathbf{v}_n\|_\infty + m \|\nabla \mathbf{v}\|_2 \\ + \|r_-\|_2 \|\nabla \mathbf{v}_n\|_\infty + m^{\gamma-2} \|r_n + r_{n+1}\|_\infty \|r_-\|_2 + \|r_-\|_2 \|\mathbf{g}\|_3 \Big) \|r\|_2 \end{aligned}$$

and by means of Young's inequality

$$m^{\gamma-1}\|r\|_2^2 \le C\Big(\frac{\|\mathbf{U}\|_3^2\|\nabla\mathbf{v}\|_2^2}{m^{\gamma-3}} + \frac{\|r_-\|_2^2\|\mathbf{U}\|_3^2\|\nabla\mathbf{v}_n\|_\infty^2}{m^{\gamma-1}} + \frac{\|\nabla\mathbf{v}\|_2^2}{m^{\gamma-3}} + \frac{\|r_-\|_2^2\|\nabla\mathbf{v}_n\|_\infty^2}{m^{\gamma-1}} + \frac{\|r_n + r_{n+1}\|_\infty^2\|r_-\|_2^2}{m^{3-\gamma}} + \frac{\|r_-\|_2^2\|\mathbf{g}\|_3^2}{m^{\gamma-1}}\Big).$$

Further using (3.95)

$$m^{\gamma-1}\|r\|_2^2 \le C(\mathbf{g})\|r_-\|_2^2\Big(\frac{1}{m^{\gamma-1}} + \frac{1}{m}\Big) + (1+E^2)\Big(C_{\mathbf{g}} + 2\frac{C_{\mathbf{g}}}{m}\Big)\|r\|_2^2. \tag{3.96}$$

The last term can be put to the left-hand side, while the rest is controlled, hence we obtain

$$\|r\|_2^2 \le \frac{C\big(\|\mathbf{g}\|_3\big)}{m^{\gamma-1}}\|r_-\|_2^2 = \frac{C_2}{m}\|r_-\|_2^2. \tag{3.97}$$

The mapping is contraction for $m > C_2$ and we can use Theorem 4.10 on the set $M_r(m) \subset W^{1,p}(\Omega)$ to get a unique solution in $M_{\mathbf{v}}(m) \times M_r(m)$. □

Elimination of the velocity linearization

We now consider (3.30)–(3.32). The last step consists in proving that mapping

$$\mathcal{T}(\mathbf{U}) = \mathbf{v}$$

possesses a fixed point, this will be proved by applying the Schauder fixed point theorem 4.8. The previous propositions yield that $\mathcal{T}$ maps $M_{\operatorname{div}\mathbf{U}}(m)$ into $M_{\mathbf{v}}(m)$. Since $M_{\mathbf{v}}(m) \subset M_{\operatorname{div}\mathbf{U}}(m)$, $M_{\operatorname{div}\mathbf{U}}(m)$ is convex and closed subset of $W^{1,\infty}(\Omega)$ and $M_{\mathbf{v}}(m)$ is compact subset of $W^{1,\infty}(\Omega)$ it remains to show that $\mathcal{T}$ is continuous on $M_{\operatorname{div}\mathbf{U}}(m)$.

Let us take $\mathbf{U}_1$, $\mathbf{U}_2$ and the corresponding solutions $(r_1, \mathbf{v}_1)$ and $(r_2, \mathbf{v}_2)$. We would like to estimate $r = r_1 - r_2$ and $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$ by means of $\mathbf{U} = \mathbf{U}_1 - \mathbf{U}_2$. We have for $k = 1, 2$

$$\begin{gathered} m\operatorname{div}\mathbf{v}_k + \operatorname{div}(r_k\mathbf{v}_k) = 0, \\ (m + r_k)\mathbf{U}_k \cdot \nabla\mathbf{v}_k - \operatorname{div}\big(2(m+r_k)\mathbb{D}(\mathbf{v}_k)\big) + \gamma m^{\gamma-1}\nabla r_k + \nabla R_m(r_k) = (m + r_k)\mathbf{g}. \end{gathered}$$

So taking the difference yields

$$\begin{aligned} m\operatorname{div}\mathbf{v} + \operatorname{div}(r\mathbf{v}_1) + \operatorname{div}(r_2\mathbf{v}) &= 0, \\ (m + r_1)\,\mathbf{U}_1 \cdot \nabla\mathbf{v} + (m + r_1)\,\mathbf{U} \cdot \nabla\mathbf{v}_2 + r\,\mathbf{U}_2 \cdot \nabla\mathbf{v}_2 - \operatorname{div}\big(2(m+r_1)\mathbb{D}(\mathbf{v})\big)& \\ - \operatorname{div}(2r\mathbb{D}(\mathbf{v}_2)) + \gamma m^{\gamma-1}\nabla r + \nabla\big(R_m(r_1) - R_m(r_2)\big) &= r\mathbf{g}. \end{aligned}$$

The standard energy estimate reads

$$\begin{aligned} (m - \|r_1\|_\infty)\|\nabla\mathbf{v}\|_2^2 \le C\big(&(m\|\mathbf{U}\|_{L^3(\Omega)}\|\nabla\mathbf{v}_2\|_2 + \|r\|_2\|\mathbf{U}_2\|_\infty\|\nabla\mathbf{v}_2\|_3)\|\mathbf{v}\|_6 \\ &+ \|r\|_2\|\mathbf{g}\|_3\|\mathbf{v}\|_6 + \|r\|_2\|\nabla\mathbf{v}_2\|_\infty\|\nabla\mathbf{v}\|_2 + m^{\gamma-2}\|\operatorname{div}\mathbf{v}_1\|_\infty\|r\|_2^2 \\ &+ m^{\gamma-2}\|\operatorname{div}(r_2\mathbf{v})\|_2\|r\|_2\big), \end{aligned}$$

where we have used the fact that the first term coming from the convective term can be rewritten

$$\int_\Omega (m + r_1)\, \mathbf{U}_1 \cdot \nabla \frac{|\mathbf{v}|^2}{2}\, \mathrm{d}x = -\int_\Omega \Big((m + r_1) \operatorname{div} \mathbf{U}_1 \frac{|\mathbf{v}|^2}{2} + \mathbf{U}_1 \cdot \nabla r_1 \frac{|\mathbf{v}|^2}{2} \Big)\, \mathrm{d}x$$

and pushed to the left-hand side, as well as the term from the nonlinear part of the pressure. Thus, after systematic usage of Young's inequality we end up with

$$\begin{aligned} m\big\|\nabla \mathbf{v}\big\|_2^2 \leq C\Big(& m\big\|\mathbf{U}\big\|_3^2\big\|\nabla \mathbf{v}_2\big\|_2^2 + \frac{\big\|r\big\|_2^2\big\|\mathbf{U}_2\big\|_\infty^2\big\|\nabla \mathbf{v}_2\big\|_3^2}{m} + \frac{\big\|r\big\|_2^2\big\|\mathbf{g}\big\|_3^2}{m} \\ & + \big\|r\big\|_2^2\big\|\nabla \mathbf{v}_2\big\|_\infty^2 + m^{\gamma-2}\big\|\operatorname{div} \mathbf{v}_1\big\|_\infty\big\|r\big\|_2^2 + m^{\gamma-2}\big\|\nabla r_2\big\|_p\big\|\nabla \mathbf{U}\big\|_2\big\|r\big\|_2\Big) \\ \leq{} & Cm\big\|\nabla \mathbf{U}\big\|_2^2 + Cm^{\gamma-2}\big\|r\big\|_2^2\big(C_{\mathbf{g}}^2 E^2 + 1\big). \end{aligned} \tag{3.98}$$

Next, we use as usually the test function $\mathbf{\Phi} = \mathcal{B}[r]$ in the momentum equation to get

$$\begin{aligned} m^{\gamma-1}\big\|r\big\|_2^2 \leq C\big(& m\,\|\mathbf{U}_1\|_3\,\|\nabla \mathbf{v}\|_2\,\|\mathbf{\Phi}\|_6 + m\,\|\mathbf{U}\|_3\,\|\nabla \mathbf{v}_2\|_2\,\|\mathbf{\Phi}\|_6 + \|r\|_2\,\big\|\mathbf{g}\big\|_3\,\|\mathbf{\Phi}\|_6 \\ & + \|r\|_2\,\|\mathbf{U}_2\|_\infty\,\|\nabla \mathbf{v}_2\|_3\,\|\mathbf{\Phi}\|_6 + 2m\,\|\nabla \mathbf{v}\|_2\,\|\nabla \mathbf{\Phi}\|_2 + \|r\|_2\,\big\|\nabla \mathbf{v}_2\big\|_\infty\,\|\nabla \mathbf{\Phi}\|_2\big) \end{aligned}$$

and using $C_{\mathbf{g}}^2, \big\|\mathbf{g}\big\|_3 \ll m^{\gamma-1}$

$$m^{\gamma-1}\,\|r\|_2^2 \leq Cm^{3-\gamma}\big(\|\mathbf{U}_1\|_3^2\,\|\nabla \mathbf{v}\|_2^2 + \|\mathbf{U}\|_3^2\,\|\nabla \mathbf{v}_2\|_2^2 + \big\|\nabla \mathbf{v}\big\|_2^2\big). \tag{3.99}$$

Combining (3.98) and (3.99) yields, using once more that $C_{\mathbf{g}}^2 E^2 \ll m$

$$m\,\|\mathbf{v}\|_{W^{1,2}(\Omega)}^2 \leq C(m, \mathbf{g})\big\|\mathbf{U}\big\|_{W^{1,2}(\Omega)}^2.$$

Moreover, we can use the higher order estimate following from the previous construction

$$m\,\|\mathbf{v}\|_{W^{2,p}(\Omega)}^2 \leq C(m, \mathbf{g})\big(\|\mathbf{U}\|_{W^{1,\infty}(\Omega)}^2 + 1\big).$$

in order to interpolate

$$\begin{aligned} \|\mathbf{v}\|_{W^{1,\infty}(\Omega)} &\leq C\,\|\mathbf{v}\|_{W^{1,2}(\Omega)}^{\beta}\,\|\mathbf{v}\|_{W^{2,p}(\Omega)}^{1-\beta} \\ &\leq C(m, \mathbf{g})\,\|\mathbf{U}\|_{W^{1,2}(\Omega)}^{\beta}\,\big(\|\mathbf{U}_1\|_{W^{1,\infty}(\Omega)}^{1-\beta} + \|\mathbf{U}_2\|_{W^{1,\infty}(\Omega)}^{1-\beta} + 1\big) \end{aligned}$$

for some $\beta \in (0, 1)$, yielding the desired continuity in $W^{1,\infty}(\Omega)$. Thus, we can apply the Schauder fixed point theorem 4.8, which completes the proof of Theorem 3.1.

Conclusion

In this thesis we have provided existence results for some problems emerging in the context of fluid flow modelling of compressible fluids. Despite of the outstanding developments of mathematical analysis of compressible flows during the last decades, some of the crucial issues of the existence and regularity of solutions to arising equations remain to be still open. Although our modest contributions to the theory do not aim to answer the most challenging questions in this field, we would like to believe that such, in some sense, purely mathematical results could clarify the limits of possible applications of those models. The knowledge of the simplifying assumptions as well as the consciousness of the discrepancy between physical reality and mathematical model should be in this context always the basic principles.

Appendix - Mathematical tools

In this chapter, we shall list the main mathematical tools used in the mathematical analysis of viscous compressible flows throughout this thesis.

4.1 Function spaces and some inequalities

The reader is assumed to be familiar with the theory of function spaces and functional analysis, which can be found in many monographs, see e.g. [16, 79, 154]. We try to follow the standard notation for function spaces, denoting the space of functions continuous on domain[3] Ω by $C(\Omega)$, and the Lebesgue spaces endowed with the norm

$$\|f\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |f|^p \,\mathrm{d}x \right)^{1/p} & \text{for } 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| & \text{for } p = \infty, \end{cases} \tag{4.100}$$

by $L^p(\Omega)$. Moreover, we denote the Sobolev spaces of functions whose m-th weak derivatives, see Schwartz [134], are integrable in the p-th power by $W^{m,p}(\Omega)$, when endowed with the norm[4]

$$\|f\|_{W^{m,p}(\Omega)} = \begin{cases} \left(\sum_{0\le|\alpha|\le m} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p} & \text{for } 1 \le p < \infty, \\ \max_{0\le|\alpha|\le m} \|D^\alpha f\|_{L^\infty(\Omega)} & \text{for } p = \infty. \end{cases} \tag{4.101}$$

We usually do not distinguish between the spaces and their vector analogues. For the most important properties of the Lebesgue and Sobolev spaces we refer again to classical monographs [2, 79, 109]. Especially, we will heavily use the reflexivity, completeness, generalized Hölder's and interpolation inequalities, trace operators, continuous and compact embeddings, the Poincaré inequality or the Green formula.

For the natural weak formulation of partial differential equations one needs to identify the dual spaces of function spaces corresponding to the solution; this is stated in the following assertion, see [2, Theorem 3.9].

Theorem 4.1 (The Dual of $W^{m,p}(\Omega)$)**.** *Let $1 \le p < \infty$, then for every $x^* \in \left(W^{m,p}(\Omega)\right)^*$, there exist functions $g_\alpha \in L^{p'}(\Omega)$ such that for all $f \in W^{m,p}(\Omega)$*

$$x^*(f) = \sum_{0\le|\alpha|\le m} \langle D^\alpha f, g_\alpha \rangle.$$

We will denote for $\frac{1}{p} + \frac{1}{p'} = 1$

$$W^{-1,p}(\Omega) \stackrel{def}{=} \left(W_0^{1,p'}(\Omega)\right)^*. \tag{4.102}$$

[3] Throughout the text, Ω denotes a domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ with sufficiently smooth boundary. We do not address here the problem of smoothness of the boundary, and state the exact requirements only in the formulation of our main theorems.

[4] The symbol α stands here for the so-called multiindex, and $|\alpha|$ for its absolute value, id est sum of components.

When dealing with time-dependent problems it is natural to introduce the spaces of functions with values in a general Banach space, namely[5]

Definition 4.2. For X a Banach space we denote by $C_{\text{weak}}(S^1, X)$ the space of all functions f ranging in X such that $y \mapsto \|f(y)\|_X$ is bounded and for every $x^* \in X^*$ the function $y \mapsto \langle x^*, f(y)\rangle_{X^*,X}$ is continuous on S^1.

Definition 4.3. For X a Banach space we denote by $L^p(S^1, X)$ the Bochner space of all strongly measurable functions being p-integrable.

From time to time, we will use abridged notation for the norms in the above defined spaces; more specifically, $\|\cdot\|_{p,q}$ means $\|\cdot\|_{L^p(S_1;L^q(\Omega))}$; while $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$. Furthermore, we usually do not distinguish between the spaces and their vectorial counterparts; since it is always clear from the context whether a certain quantity is a vector or not.

Theorem 4.4 (Korn). *Let $1 < p < +\infty$, $d \geq 2$, then there exists a constant $c = c(p, d, \Omega)$ such that*

1. *for all $\mathbf{f} \in W_0^{1,p}(\Omega)$*

$$\|\nabla \mathbf{f}\|_{L^p(\Omega)} \leq c \Big\|\nabla \mathbf{f} + \nabla^T \mathbf{f} - \frac{2}{d} \operatorname{div} \mathbf{f}\, \mathbb{I}\Big\|_{L^p(\Omega)}$$

2. *if Ω has no axial symmetry then for all $\mathbf{f} \in W^{1,p}(\Omega)$ with $\mathbf{f} \cdot \mathbf{n} = \mathbf{0}$ on $\partial\Omega$*

$$\|\mathbf{f}\|_{W^{1,p}(\Omega)} \leq c \Big\|\nabla \mathbf{f} + \nabla^T \mathbf{f} - \frac{2}{d} \operatorname{div} \mathbf{f}\, \mathbb{I}\Big\|_{L^p(\Omega)}$$

3. *if $\Omega \subset \mathbb{R}^3$ then for all $\mathbf{f} \in W^{1,p}(\Omega)$*

$$\|\mathbf{f}\|_{W^{1,p}(\Omega)} \leq c \Big(\Big\|\nabla \mathbf{f} + \nabla^T \mathbf{f} - \frac{2}{d} \operatorname{div} \mathbf{f}\, \mathbb{I}\Big\|_{L^p(\Omega)} + \int_\Omega |\mathbf{f}| \, \mathrm{d}x \Big)$$

For the proofs see e.g. [48, Theorem 10.16], see also [20, 29, 70].

4.2 Compactness

Let us begin this section with the consequences of compactness within the space of continuous functions. We have the following corollary of an abstract version of the Arzelà–Ascoli theorem, see [79]

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, let $d \geq 2$, and let $1 < p, q < \infty$ and $\{g_n\}_{n=1}^\infty$ be a sequence of functions $g_n \in C_{\text{weak}}(S^1, L^q(\Omega))$ such that g_n are uniformly bounded in $L^q(\Omega)$ and uniformly continuous in $W^{-1,p}(\Omega)$. Then we have (up to subsequence)*

$$g_n \to g \text{ in } C_{\text{weak}}(S^1, L^q(\Omega)).$$

Furthermore, if L^q is compactly embedded into $W^{-1,p}(\Omega)$, then we have even

$$g_n \to g \text{ in } C(S^1, W^{-1,p}(\Omega)).$$

[5]For the purpose of formulation of the time-periodic problem we use the notation $S^1 = [0, L]|_{\{0,L\}}$ for the time interval accompanied with the periodicity condition $g(0, .) = g(L, .)$.

For the proof see e.g. [118, Lemma 6.2].

Based on the Arzelà–Ascoli argument are also the well-known results for compactness in the Lebesgue (Kolmogorov's theorem) and the Sobolev spaces (Rellich–Kondrachov's theorem), as well as theorem concerning the compactness of the trace operator. We refer the interested reader to the monographs [2, 79].

A useful tool which combines the compactness in time and in space is the eminent Aubin–Lions lemma [5].

Lemma 4.6. *Let X_1, X_2, X be Banach spaces such that $X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$.*[6] *Suppose further that X_0, X_1 are reflexive and that $1 < p, q < \infty$. Then the space $Y = \{u \in L^p(S^1, X_0), \partial_t u \in L^q(S^1, X_1)\}$ is compactly embedded into $L^p(S^1, X)$,*

$$Y \hookrightarrow\hookrightarrow L^p(S^1, X).$$

For a proof of a more general version see e.g. Roubíček [133, Lemma 7.7].

Let us conclude this section with some fixed point results involving compactness, see [16, 154].

Theorem 4.7 (Brouwer)**.** *Let $K \subset \mathbb{R}^d$ be non-empty convex and compact set, and let $\mathcal{T}$ be a continuous mapping $\mathcal{T}: K \to K$. Then $\mathcal{T}$ possesses a fixed point.*

More generally, in infinite dimensions we have

Theorem 4.8 (Schauder)**.** *Let X be a Banach space, $K \subset X$ be non-empty convex and compact set, let $\mathcal{T}$ be a continuous mapping $\mathcal{T}: K \to K$. Then $\mathcal{T}$ possesses a fixed point.*

The following consequence of the Schauder theorem is sometimes more suitable for applications.

Theorem 4.9 (Leray–Schauder)**.** *Let X be a Banach space, and $\mathcal{T}$ a continuous and compact mapping $\mathcal{T}: X \to X$, such that the possible fixed points $x = \lambda \mathcal{T} x$, $0 \leq \lambda \leq 1$ are bounded in X. Then $\mathcal{T}$ possesses a fixed point.*

Without the compactness, the existence of fixed point can be still guaranteed provided we deal with contraction.

Theorem 4.10 (Banach)**.** *Let X, Y be Banach spaces, such that X is reflexive and continuously embedded into Y ($X \hookrightarrow Y$), let $K \subset X$ be a non-empty, convex, bounded subset of X. Suppose further that $\mathcal{T}: K \to K$ is a contraction mapping in Y-metric, id est*

$$\|\mathcal{T}(u) - \mathcal{T}(v)\|_Y \leq \varkappa \|u - v\|_Y, \quad \forall u, v \in K,$$

for some $0 \leq \varkappa < 1$. Then $\mathcal{T}$ possesses a unique fixed point in K.

For the proof, which is based on the Banach contraction principle combined with Theorem 4.16, see e.g. [130, Theorem 0.1].

[6] We use the notation $A \hookrightarrow B$ and $A \hookrightarrow\hookrightarrow B$ for A being continuously, or compactly embedded into B, respectively.

4.3 Regularity of some equations

The regularity of partial differential equations is discussed in many monographs, see e.g. Nečas [109], Evans [32], Roubíček [133]. We will begin this section with L^p-regularity properties of an elliptic equation in a divergence form with Neumann boundary condition

$$\begin{aligned} -\operatorname{div}(A(x)\nabla u) + c(x)u = g \text{ in } \Omega, \\ A\nabla u \cdot \mathbf{n} = h \text{ on } \partial\Omega, \end{aligned} \tag{4.103}$$

where we assume A to be elliptic $\left(\sum_{i,j} a_{ij}\xi_i\xi_j \geq \alpha\,|\xi|^2,\ \forall \xi \in \mathbb{R}^d\right)$ with some $\alpha > 0$ and symmetric $(a_{ij} = a_{ji})$. The following result due to Agmon, Douglis, Nirenberg [3] holds true.

Theorem 4.11. *Let $\Omega \in \mathbb{R}^d$ be a bounded domain with C^2 boundary, $1 < p < \infty$, assume that $A \in C^1(\overline{\Omega})$, $c \in C(\overline{\Omega})$, $g \in L^p(\Omega)$, $h \in W^{1-1/p,p}(\partial\Omega)$. Then there exists $C > 0$ such that any solution to (4.103) satisfies*

$$\|u\|_{W^{2,p}(\Omega)} \leq C\left(\|g\|_{L^p(\Omega)} + \|h\|_{W^{1-1/p,p}(\partial\Omega)} + \|u\|_{L^p(\Omega)}\right). \tag{4.104}$$

Remark 4.12. Theorem 4.11 remains true if Ω is chosen to be torus in one of the directions $(S^1 \times \Omega)$, furthermore the last term on the right-hand side of the estimate (4.104) may be replaced by $\|u\|_{L^p(\widetilde{\Omega})}$, where $\widetilde{\Omega} \subset \Omega$ has a positive measure.

Theorem 4.13 (Bogovskii)**.** *Let $\Omega \in \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, then there exists a bounded linear operator $\mathcal{B}$*

$$\mathcal{B} : \left\{ f, f \in L^p(\Omega), \int_\Omega f(x)\,\mathrm{d}x = 0 \right\} \to W_0^{1,p}(\Omega),\ 1 < p < \infty$$

such that

1. $\operatorname{div}(\mathcal{B}[f]) = f$ *a.e. in Ω, id est $\mathbf{u} = \mathcal{B}[f]$ solves the equation* $\operatorname{div}\mathbf{u} = f$ *with the boundary condition $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$*

2. *there exists a constant $c = c(d, p, \Omega)$ such that*

$$\|\mathcal{B}[f]\|_{W^{1,p}(\Omega)} \leq c\,\|f\|_{L^p(\Omega)},\ \forall 1 < p < \infty.$$

The proof can be found in many monographs on mathematical analysis of fluid dynamics [48, 118]; for some generalizations see also Danchin, Mucha [21, 22].

We will also need the regularity properties of the solutions to the following (overdetermined) system

$$\begin{aligned} \operatorname{curl}\mathbf{u} = \mathbf{f} \text{ in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega, \\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega, \end{aligned} \tag{4.105}$$

with the compatibility conditions $\operatorname{div}\mathbf{f} = 0$ in Ω, $\mathbf{f}\cdot\mathbf{n} = 0$ on $\partial\Omega$. The following result holds true, see Solonnikov [140]; the assumption concerning the regularity of the domain can be relaxed, see Mucha, Pokorný [103].

Theorem 4.14. *Let $\Omega \in \mathbb{R}^3$ be a bounded domain with Lipschitz boundary, let $\mathbf{f} \in W^{1,p}(\Omega)$, $1 < p < +\infty$, $\operatorname{div} \mathbf{f} = 0$ in Ω, $\mathbf{f} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Then there exists a constant $c = c(\Omega, p)$ such that the unique solution $\mathbf{u}$ to system (4.105) satisfies*

$$\|\nabla \mathbf{u}\|_{L^p(\Omega)} \leq c \|\mathbf{f}\|_{L^p(\Omega)}, \qquad \left\|\nabla^2 \mathbf{u}\right\|_{L^p(\Omega)} \leq c \|\mathbf{f}\|_{W^{1,p}(\Omega)}.$$

In the forthcoming sections, we will recall some tools of the compensated compactness, developed by Tartar [143], DiPerna [26], Lions [85], Feireisl and others [50]. They enable us to overcome in certain situations the problem of convergence in nonlinear terms involving weakly convergent subsequences with the lack of standard compactness techniques.

4.4 Weak convergence

In a theory of partial differential equations, it is often convenient to approximate the original problem in a suitable way and then consider the corresponding limit passage. The following assertions appear to be extremely useful in this context.

Theorem 4.15 (Banach–Alaoglu)**.** *In a Banach space with a separable predual, any bounded sequence contains a weakly* convergent subsequence.*

Theorem 4.16 (Kakutani)**.** *Let X be a reflexive Banach space, then the ball B_X is weakly compact.*

On the other hand, it is well known that for a nonlinear function P and weakly convergent sequence u_n it is generally not necessarily true that $P(u) = \overline{P(u)}$.[7] However, as we will now recall, certain additional information can be obtained as soon as we deal with monotone or convex functions. The following consequence of the Minty trick is taken from [48, Theorem 10.19], see also [118, Section 3.4.2].

Lemma 4.17. *Let $\Omega \subset \mathbb{R}^n$, $I \subset \mathbb{R}$ be an interval and $P, Q : I \mapsto \mathbb{R}$ be two continuous nondecreasing functions. Let $u_n \in L^1(\Omega; I)$ be a sequence of functions such that $P(u_n) \rightharpoonup \overline{P(u)}$, $Q(u_n) \rightharpoonup \overline{Q(u)}$, $P(u_n) \cdot Q(u_n) \rightharpoonup \overline{P(u)Q(u)}$ weakly in $L^1(\Omega; I)$. Then*

(i) $\overline{P(u)}\,\overline{Q(u)} \leq \overline{P(u)Q(u)}$,

(ii) if additionally $I = \mathbb{R}$, $Q(\mathbb{R}) = \mathbb{R}$, Q is strictly increasing and $\overline{P(u)}\,\overline{Q(u)} = \overline{P(u)Q(u)}$, then
$$\overline{P(u)} = P\Big(Q^{-1}\big(\overline{Q(u)}\big)\Big),$$

(iii) especially for $Q(z) = z$ we have $\overline{P(u)} = P(u)$.

Convex lower semi-continuous functionals are weakly lower semi-continuous, moreover the following statements according to [48, Theorem 10.20] concerning the weak convergence and convexity holds true as well, see also [30].

[7] Recall that we denote a weak limit of nonlinear expressions $\{P(u_n)\}$ by $\overline{P(u)}$.

Theorem 4.18. *Let $\Omega \subset \mathbb{R}^d$ be a measurable set, let $\{\mathbf{u}_n\}_{n=1}^{\infty} \subset L^1(\Omega, \mathbb{R}^N)$ be such that*

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \text{ in } L^1(\Omega, \mathbb{R}^N).$$

Suppose further that there exists a lower semi-continuous convex function $\Phi : \mathbb{R}^N \to (-\infty, +\infty]$ such that $\Phi(\mathbf{u}_n) \in L^1(\Omega)$ for all n, and

$$\Phi(\mathbf{u}_n) \rightharpoonup \overline{\Phi(\mathbf{u})} \text{ in } L^1(\Omega, \mathbb{R}^N).$$

Then

$$\Phi(\mathbf{u}) \leq \overline{\Phi(\mathbf{u})} \text{ a.e. in } \Omega.$$

Moreover, if Φ is even strictly convex on an open convex $U \subset \mathbb{R}^N$, and

$$\Phi(\mathbf{u}) = \overline{\Phi(\mathbf{u})} \text{ a.e. in } \Omega,$$

then (up to a subsequence)

$$\mathbf{u}_n(\mathbf{x}) \to \mathbf{u}(\mathbf{x}) \text{ for a.a. } \mathbf{x} \in \{\mathbf{x} \in \Omega, \mathbf{u}(\mathbf{x}) \in U\}.$$

4.5 Div-Curl lemma of compensated compactness

We begin this section with the famous Helmholtz–Weyl decomposition, see e.g. Galdi [56, Section III.1].

Theorem 4.19 (Helmholtz decomposition)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $C^{1,1}$-boundary. Then for any $1 < p < \infty$ the space $L^p(\Omega, \mathbb{R}^d)$ can be decomposed as follows*

$$L^p(\Omega, \mathbb{R}^d) = L^p_{0,\mathrm{div}}(\Omega, \mathbb{R}^d) \oplus L^p_g(\Omega, \mathbb{R}^d),$$

where $L^p_{0,\mathrm{div}}(\Omega, \mathbb{R}^d) = \big\{\mathbf{u} \in L^p(\Omega, \mathbb{R}^d),\ \mathrm{div}\,\mathbf{u} = 0 \text{ in } \Omega,\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\big\}$, and $L^p_g(\Omega, \mathbb{R}^d) = \big\{\mathbf{u} \in L^p(\Omega, \mathbb{R}^d),\ \mathbf{u} = \nabla\psi \text{ in } \Omega,\ \psi \in W^{1,p}_{\mathrm{loc}}(\Omega)\big\}$.

We will use linear operators, induced by the previous theorem,

$$P_{\triangledown} : L^p(\Omega) \to W^{1,p}(\Omega) \qquad \text{and} \qquad P_H : L^p(\Omega) \to L^p_{\mathrm{div}}(\Omega) \tag{4.106}$$

with properties $\mathbf{g} = P_H(\mathbf{g}) + \nabla P_{\triangledown}(\mathbf{g})$, $\mathrm{div}\,\mathbf{g} = \Delta P_{\triangledown}(\mathbf{g})$, $\mathrm{curl}\, P_H\mathbf{g} = \mathrm{curl}\,\mathbf{g}$ in Ω and $\mathbf{n}\cdot P_H(\mathbf{g}) = 0$ on $\partial\Omega$.

As Tartar [143] and Murat [107] observed, this decomposition enables one to prove a powerful compensated compactness result — the celebrated Div-Curl Lemma, see e.g. [48, Theorem 10.21].

Lemma 4.20. *Let $\mathbf{U}_\delta \rightharpoonup \mathbf{U}$ in $L^p(\mathbb{R}^N, \mathbb{R}^N)$, $\mathbf{V}_\delta \rightharpoonup \mathbf{V}$ in $L^q(\mathbb{R}^N, \mathbb{R}^N)$, with*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{s} < 1.$$

Suppose further that $\mathrm{div}\,\mathbf{V}_\delta$ is precompact in $W^{-1,r}(\mathbb{R}^N, \mathbb{R})$, and that $\mathrm{curl}\,\mathbf{V}_\delta$ is precompact in $W^{-1,r}(\mathbb{R}^N, \mathbb{R}^{N\times N})$ for some $r \in (1,\infty)$.[8] Then

$$\mathbf{V}_\delta \cdot \mathbf{U}_\delta \rightharpoonup \mathbf{V}\cdot\mathbf{U} \text{ in } L^s(\mathbb{R}^N).$$

[8]Note that the operators div and curl represent here their N-dimensional versions in contrast to the rest of the thesis where they are used in their usual d-dimensional sense.

We will also exploit some psudodifferential operators defined on whole $\mathbb{R}^d$ by means of the Fourier transform, namely inverse divergence $\nabla\Delta^{-1}$

$$\partial_j \Delta^{-1}[v] = \mathcal{F}^{-1}\left[\frac{\mathrm{i}\xi_j}{|\xi|^2}\mathcal{F}(v)(\xi)\right], \tag{4.107}$$

and the "double" Riesz transform $\mathcal{R}$

$$\mathcal{R}_{ij}[v] = \mathcal{F}^{-1}\left[\frac{\xi_i\xi_j}{|\xi|^2}\mathcal{F}(v)(\xi)\right], \tag{4.108}$$

which is according to the Calderón-Zygmund theory a bounded operator on any $L^p(\Omega)$, $1 < p < \infty$. The following two commutators lemma involving the Riesz transformation are consequences of Lemma 4.20 and the Riesz–Thorin interpolation theorem, see e.g. [48, Section 10.17].

Lemma 4.21. *Let $\mathbf{V}_\delta \rightharpoonup \mathbf{V}$ in $L^p(\mathbb{R}^3, \mathbb{R}^3)$, and $w_\delta \rightharpoonup w$ in $L^q(\mathbb{R}^3)$, with*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{s} < 1.$$

Then

$$w_\delta \mathcal{R}[\mathbf{V}_\delta] - \mathcal{R}[w_\delta]\mathbf{V}_\delta \rightharpoonup w\mathcal{R}[\mathbf{V}] - \mathcal{R}[w]\mathbf{V} \text{ in } L^s(\mathbb{R}^3, \mathbb{R}^3).$$

Lemma 4.22. *Let $\mathbf{V} \in L^p(\mathbb{R}^3, \mathbb{R}^3)$, and $w \in W^{1,q}(\mathbb{R}^3)$, where $r \in (1, 3)$, $p \in (1, \infty)$,*

$$\frac{1}{p} + \frac{1}{q} - \frac{1}{3} < \frac{1}{s} < 1.$$

Then

$$\|\mathcal{R}[w\mathbf{V}] - w\mathcal{R}[\mathbf{V}]\|_{W^{a,s}(\mathbb{R}^3)} \le C\,\|w\|_{W^{1,r}(\mathbb{R}^3)}\,\|\mathbf{V}\|_{L^q(\mathbb{R}^3)},$$

with $\frac{a}{3} = \frac{1}{s} + \frac{1}{3} - \frac{1}{p} - \frac{1}{q}$; $W^{a,s}(\mathbb{R}^3)$ denotes the Sobolev–Slobodetskii space.

4.6 Renormalized continuity equation and oscillations defect measure

Suppose that $(\varrho, \mathbf{v})$ satisfies the continuity equation

$$\frac{\partial\varrho}{\partial t} + \operatorname{div}(\varrho\mathbf{v}) = 0, \tag{4.109}$$

if we formally multiply (4.109) by $b'(\varrho)$ for $b \in C^1((0,\infty))$ we end up with the renormalized continuity equation

$$\frac{\partial\big(b(\varrho)\big)}{\partial t} + \operatorname{div}\big(b(\varrho)\mathbf{v}\big) + \big(\varrho b'(\varrho) - b(\varrho)\big)\operatorname{div}\mathbf{v} = 0. \tag{4.110}$$

The steady version of (4.110) reads

$$\operatorname{div}\big(b(\varrho)\mathbf{v}\big) + \big(\varrho b'(\varrho) - b(\varrho)\big)\operatorname{div}\mathbf{v} = 0. \tag{4.111}$$

As we should see, the fact that the solutions of the original problem solves also the renormalized version of the continuity equation for some b's is one of the crucial tools in the mathematical theory of compressible flows. If the density is square integrable, which is the case in Chapter 2, this can be obtained directly thanks to a regularization method due to DiPerna and Lions. Namely, the following variant of Fridrichs' commutator lemma holds true, see DiPerna, Lions [26, Lemma II.1], Novotný, Straškraba [118, Lemma 3.1]

Lemma 4.23. *Let $d \geq 2$, let $1 < p,\ q < \infty$, $\frac{1}{p} + \frac{1}{q} \leq 1$ and suppose that*

$$\varrho \in L^q_{\text{loc}}(\mathbb{R}^d), \qquad \mathbf{v} \in W^{1,p}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d).$$

Then for $\frac{1}{r} \geq \frac{1}{p} + \frac{1}{q}$ we have[9]

$$S_\epsilon(\mathbf{v} \cdot \nabla \varrho) - \mathbf{v} \cdot \nabla S_\epsilon(\varrho) \to 0, \text{ strongly in } L^r_{\text{loc}}(\mathbb{R}^d),$$

where S_ϵ is a standard mollifier.

As a consequence we obtain in particular for the steady case the following lemma, see [118, Lemma 3.3]

Lemma 4.24. *Let $d \geq 2$, $2 \leq \beta < \infty$, $\lambda_0 < 1$, $-1 < \lambda_1 \leq \frac{\beta}{2} - 1$, $b \in C([0,\infty)) \cap C^1((0,\infty))$ satisfying*

$$|b'(t)| \leq C t^{-\lambda_0}, \quad \text{for } t \in [0,1], \tag{4.112}$$

$$|b'(t)| \leq C t^{\lambda_1}, \quad \text{for } t \geq 1. \tag{4.113}$$

Suppose that $\varrho \in L^\beta_{\text{loc}}(\mathbb{R}^d)$, $\varrho \geq 0$ a.e. in $\mathbb{R}^d$, $\mathbf{v} \in W^{1,2}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d)$ solve $\operatorname{div}(\varrho \mathbf{v}) = 0$ in distributional sense in $\mathbb{R}^d$. Then (4.111) holds true for any b satisfying growth conditions (4.112)–(4.113).

In the general case, one has to use more sophisticated tools. Inspired by Jiang and Zhang [72], Feireisl [35,50] introduced for a weakly convergent sequence $\varrho_\delta \rightharpoonup \varrho$ the so-called oscillations defect measure, see also [48, Section 3.7.5],

$$\mathbf{osc_q}[\varrho_\delta \to \varrho](S^1 \times \Omega) := \sup_{k>0} \limsup_{\delta \to 0+} \int_{S^1} \int_\Omega |T_k(\varrho_\delta) - T_k(\varrho)|^q \, \mathrm{d}x \, \mathrm{d}t, \tag{4.114}$$

where T_k is defined as a smooth concave function such that

$$T_k(z) = kT\left(\frac{z}{k}\right), \quad T(z) = \begin{cases} z \text{ for } z \in [0,1], \\ 2 \text{ for } z \in [3,\infty). \end{cases}$$

We have the following [48, Lemma 3.8].

Lemma 4.25. *Let $\Omega \subset \mathbb{R}^3$ be open, and assume that we have a family of distributional solutions $(\varrho_\delta, \mathbf{v}_\delta)$ to renormalized continuity equation (4.110) for any $b \in C^\infty[0,\infty)$, $b' \in C^\infty_c[0,\infty)$, such that for some $r > 1$*

$$\varrho_\delta \rightharpoonup \varrho \text{ in } L^1(S^1 \times \Omega), \tag{4.115}$$

$$\mathbf{v}_\delta \rightharpoonup \mathbf{v} \text{ in } L^r(S^1 \times \Omega), \tag{4.116}$$

$$\nabla \mathbf{v}_\delta \rightharpoonup \nabla \mathbf{v} \text{ in } L^r(S^1 \times \Omega). \tag{4.117}$$

[9] The first term has to be interpreted in the sense of distributions $\mathbf{v} \cdot \nabla \varrho = \operatorname{div}(\varrho \mathbf{v}) - \varrho \operatorname{div} \mathbf{v}$

Suppose further that for $\frac{1}{q} < 1 - \frac{1}{r}$

$$\mathbf{osc_q}[\varrho_\delta \to \varrho](S^1 \times \Omega) < +\infty.$$

Then the limit functions $(\varrho, \mathbf{v})$ *satisfy in the distributional sense the renormalized continuity equation (4.110) for all* $b \in C^1[0,\infty) \cap W^{1,\infty}(0,\infty)$.

This result can be extended by means of the Lebesgue dominated convergence theorem up to $b \in C([0,\infty)) \cap C^1((0,\infty))$ with growth conditions (4.112)–(4.113).

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